MATH253X-UX1 Final Exam Name: Answer Key
Spring 2019

Instructions. You have 120 minutes. Closed book, closed notes, no calculator. *Show all your work* in order to receive full credit.

1. Consider the following points in space: $A(-2,0,1)$, $B(1,1,-1)$, and $C(0,2,0)$.

(a) Find parametric equations for the line going through A and B.

Solution:

$$\overrightarrow{AB} = \langle 1+2, 1-0, -1-1\rangle = \langle 3,1,-2\rangle$$

So parametric equations for the line are:

$$\boxed{\begin{cases} x = -2+3t \\ y = t \\ z = 1-2t \end{cases}}$$

(b) Find the area of the parallelogram with adjacent sides AB and AC.

Solution: Let A be the area of the parallelogram.

$$\overrightarrow{AC} = \langle 0+2, 2-0, 0-1\rangle = \langle 2,2,-1\rangle$$

$$\overrightarrow{AB}\times\overrightarrow{AC} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -2 \\ 2 & 2 & -1 \end{vmatrix} = \langle -1+4, -4+3, 6-2\rangle = \langle 3,-1,4\rangle$$

$$\Rightarrow \quad A = \left\|\overrightarrow{AB}\times\overrightarrow{AC}\right\| = \sqrt{3^2+(-1)^2+4^2} = \boxed{\sqrt{26}}$$

2. Assume a particle has velocity $\mathbf{v}(t) = \langle 2t, t^2, 2\rangle$ with speed measured in ft/s.

(a) Find the position vector $\mathbf{r}(t)$ at all times if $\mathbf{r}(2) = \langle 2,3,1\rangle$.

Solution:

$$\begin{aligned}
\mathbf{r}(t) - \mathbf{r}(2) &= \int_2^t \mathbf{v}(u)\,du = \int_2^t \langle 2u, u^2, 2\rangle\,du = \left[\left\langle u^2, \frac{u^3}{3}, 2u\right\rangle\right]_2^t \\
\Rightarrow \quad \mathbf{r}(t) - \langle 2,3,1\rangle &= \left\langle t^2, \frac{t^3}{3}, 2t\right\rangle - \left\langle 4, \frac{8}{3}, 4\right\rangle \\
\Rightarrow \quad \mathbf{r}(t) &= \left\langle t^2, \frac{t^3}{3}, 2t\right\rangle + \left\langle 2-4, 3-\frac{8}{3}, 1-4\right\rangle \\
&= \left\langle t^2, \frac{t^3}{3}, 2t\right\rangle + \left\langle -2, \frac{1}{3}, -3\right\rangle \\
\Rightarrow \quad & \boxed{\mathbf{r}(t) = \left\langle t^2-2, \frac{t^3+1}{3}, 2t-3\right\rangle}
\end{aligned}$$

(b) Find the distance traveled from $t=1$ s to $t=3$ s.

Solution: Let d be the distance traveled.

$$\mathbf{v}(t) = \langle 2t, t^2, 2\rangle \quad \Rightarrow \quad \|\mathbf{v}(t)\| = \sqrt{4t^2+t^4+4} = \sqrt{(t^2+2)^2} = t^2+2$$

$$\Rightarrow \quad d = \int_1^3 \|\mathbf{v}(t)\|\,dt = \int_1^3 t^2+2\,dt = \left[\frac{t^3}{3}+2t\right]_1^3 = 9+6-\frac{1}{3}-2 = \boxed{\frac{38}{3}\text{ ft}}$$

3. Let $f(x,y) = x^2y^2 - xy^2 - x^2 - 2y^2 + x$.

(a) Verify that $(1/2, 0)$ and $(-1, 1)$ are (among the) critical points of $f(x,y)$. Then classify them using the Second Partials Test.

Solution: We have:

$$\begin{aligned} \nabla f(x,y) &= \left\langle 2xy^2 - y^2 - 2x + 1, 2x^2y - 2xy - 4y \right\rangle \\ \Rightarrow \quad \nabla f(1/2, 0) &= \langle 0 - 0 - 1 + 1, 0 - 0 - 0 \rangle = \langle 0, 0 \rangle \qquad \checkmark \\ \Rightarrow \quad \nabla f(-1,1) &= \langle 2(-1)(1) - 1 + 2 + 1, 2(1)(1) - 2(-1)(1) - 4(1) \rangle = \langle 0, 0 \rangle \qquad \checkmark \end{aligned}$$

So they are indeed critical points and:

$$\begin{aligned} & f_{xx} = 2y^2 - 2 \quad , \quad f_{yy} = 2x^2 - 2x - 4 \quad , \quad f_{xy} = 4xy - 2y \\ \Rightarrow \quad & d(x,y) = f_{xx}f_{yy} - f_{xy}^2 = 4(y^2 - 1)(x^2 - x - 2) - 4(2xy - y)^2 \end{aligned}$$

- $d(1/2, 0) = 4(-1)\left(\frac{1}{4} - \frac{1}{2} - 2\right) - 4(0) = 9 > 0$, $f_{xx}(1/2, 0) = -2 < 0$ so $\boxed{(1/2, 0) \text{ is a local maximum}}$;
- $d(-1,1) = 4(0)(1 + 1 - 2) - 4(-2-1)^2 = -36 < 0$, so $\boxed{(-1, 1, -2) \text{ is a saddle point}}$;

(b) Find the directional derivative of f when moving from $(0, 2)$ towards $(-1, 3)$.

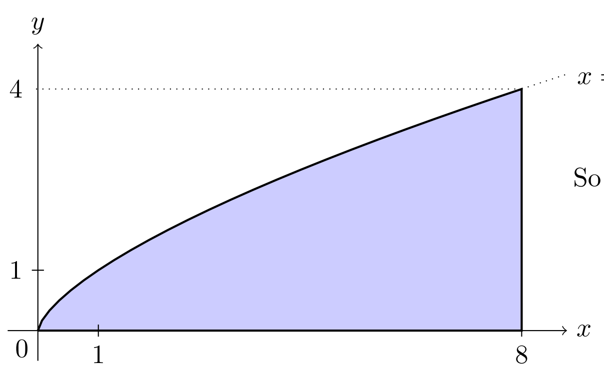
Solution:

$$\begin{aligned} \nabla f(x,y) &= \left\langle 2xy^2 - y^2 - 2x + 1, 2x^2y - 2xy - 4y \right\rangle \\ \Rightarrow \quad \nabla f(0,2) &= \langle 0 - 4 - 0 + 1, 0 - 0 - 8 \rangle = \langle -3, -8 \rangle \\ \mathbf{v} = \langle -1 - 0, 3 - 2 \rangle = \langle -1, 1 \rangle \quad &\Rightarrow \quad \mathbf{u} = \frac{\mathbf{v}}{\|\mathbf{v}\|} = \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle \\ D_{\mathbf{u}}f(0,2) = \nabla f(0,2) \cdot \mathbf{u} &= \langle -3, -8 \rangle \cdot \left\langle -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right\rangle = \frac{3}{\sqrt{2}} - \frac{8}{\sqrt{2}} = \boxed{-\frac{5\sqrt{2}}{2}} \end{aligned}$$

4. Switch the order of integration then compute

$$I = \int_0^4 \int_{y^{\frac{3}{2}}}^8 \sqrt{y}e^{x^2}\, dx\, dy$$

Solution: The region of integration is:



$x = y^{\frac{2}{3}}$ or $y = x^{\frac{3}{2}}$

So switching the order of integration we have:

$$\boxed{I = \int_0^8 \int_0^{x^{\frac{2}{3}}} \sqrt{y}e^{x^2}\, dy\, dx}$$

And we compute:

$$I = \int_0^8 \int_0^{x^{\frac{2}{3}}} \sqrt{y}e^{x^2}\, dy\, dx = \int_0^8 \left[\frac{2y^{\frac{3}{2}}}{3} e^{x^2} \right]_0^{x^{\frac{2}{3}}} dx = \int_0^8 \frac{2x}{3} e^{x^2} - 0\, dx = \left[\frac{e^{x^2}}{3} \right]_0^8 = \boxed{\frac{e^{64} - 1}{3}}.$$

5. Consider a particle moving along C parametrized by $\mathbf{r}(t) = \langle t^2 - 1, 2t, t\rangle$, $1 \le t \le 2$ through the vector field $\mathbf{F}(x, y, z) = \langle 2xy - 1, x^2 - z, 2z - y\rangle$.

(a) The field is conservative. Find *all* potential functions.

Solution: We have that for any potential function f, $\mathbf{F}(x, y, z) = \langle P, Q, R\rangle = \langle f_x, f_y, f_z\rangle$. So,

$$f(x,y,z) = \int P\,dx = \int 2xy - 1\,dx = x^2y - x + C_1(y,z)$$
$$f(x,y,z) = \int Q\,dy = \int x^2 - z\,dy = x^2y - yz + C_2(x,z)$$
$$f(x,y,z) = \int R\,dx = \int 2z - y\,dz = z^2 - yz + C_3(x,y)$$
$$\Rightarrow \boxed{f(x,y,z) = x^2y - x - yz + z^2 + C}$$

(b) Apply the Fundamental Theorem of Line Integrals to compute the circulation (work).

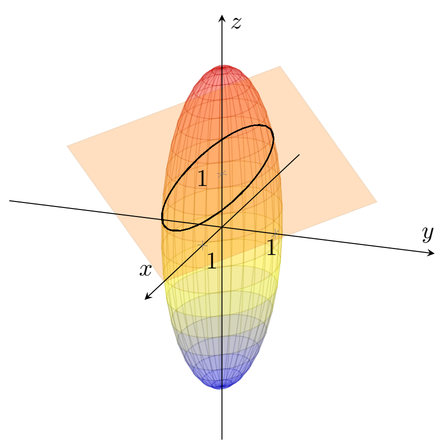
Solution:

$$\mathbf{r}(1) = \langle 0, 2, 1\rangle \quad , \quad \mathbf{r}(2) = \langle 3, 4, 2\rangle$$
$$W = \int_C \mathbf{F}\cdot d\mathbf{r} = f(3,4,2) - f(0,2,1)$$
$$= 9(4) - 3 - 4(2) + 4 - (0 - 0 - 2(1) + 1) = 29 - (-1) = \boxed{30}$$

6. Sketch the following:

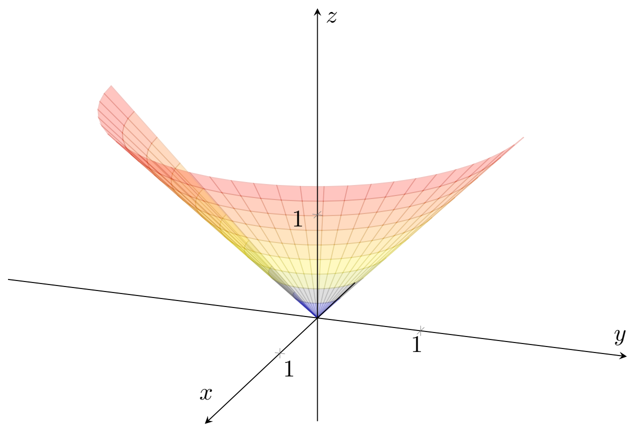
(a) the surfaces $4x^2 + 9y^2 + z^2 = 9$ and $2x - 3y + 6z = 6$ and their intersection;

Solution: We have an ellipsoid and a plane, so their intersection is elliptic in shape.



(b) the surface given in spherical coordinates by $\phi = \frac{\pi}{4}$, $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, and $0 \le \rho \le 2\sec\phi$.

Solution: The surface $\phi = \frac{\pi}{4}$ is the cone $z = \sqrt{x^2 + y^2}$, but only for $x \ge 0$ from the θ restriction, and for the restriction in ρ, we can rewrite $0 \le \rho \le 2\sec\phi$ as $0 \le \rho\cos\phi \le 2$ that is $0 \le z \le 2$.



7. Consider the hyperboloid of two sheets:

$$x^2 + 4y^2 - z^2 = -4$$

(a) Find an equation of the tangent plane to the hyperboloid at $(1, -1, 3)$.

Solution: Let $F(x, y, z) = x^2 + 4y^2 - z^2 = -4$. Then,

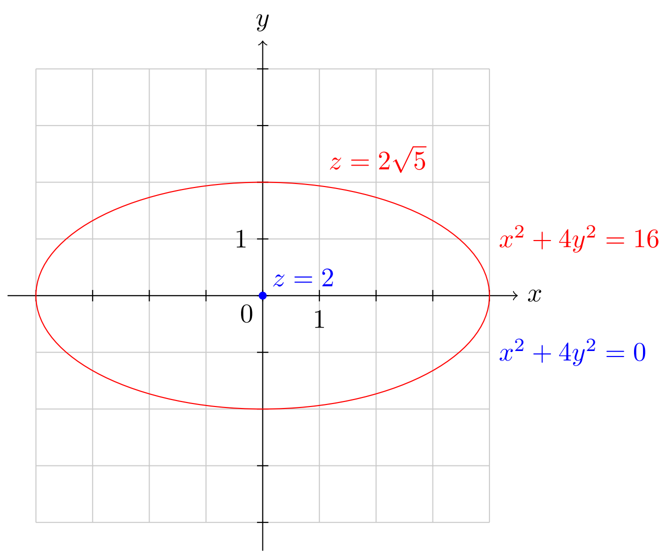
$$\nabla F(x, y, z) = \langle 2x, 8y, -2z\rangle \quad \Rightarrow \quad \nabla F(1, -1, 3) = \langle 2, -8, -6\rangle .$$

And so the equation of the tangent plane is:

$$2(x-1) - 8(y+1) - 6(z-3) = 0 \quad \text{or} \quad \boxed{x - 4y - 3z + 4 = 0} .$$

(b) Sketch the level curves corresponding to $z = 2$ and $z = 2\sqrt{5}$.

Solution:

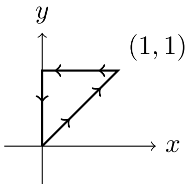


(c) Fully SET UP an expression with triple integrals to represent $\bar{x}$ in the center of mass of the solid bounded by the hyperboloid and the plane $z = 2\sqrt{5}$ if the density of the solid is given by $\rho(x, y, z) = 2y^2 z$. DO NOT EVALUATE.

Solution:

$$\boxed{\bar{x} = \frac{\displaystyle\int_{-4}^{4}\int_{-\frac{\sqrt{16-x^2}}{2}}^{\frac{\sqrt{16-x^2}}{2}}\int_{\sqrt{x^2+4y^2+4}}^{2\sqrt{5}} 2xy^2z\;dz\;dy\;dx}{\displaystyle\int_{-4}^{4}\int_{-\frac{\sqrt{16-x^2}}{2}}^{\frac{\sqrt{16-x^2}}{2}}\int_{\sqrt{x^2+4y^2+4}}^{2\sqrt{5}} 2y^2z\;dz\;dy\;dx}}$$

8. Use Green's theorem to find the circulation of the vector field $\mathbf{F}(x,y) = \langle ye^x - \sin x, 2xy\rangle$ over the closed curve C described below:



Solution: We verify that C is oriented counterclockwise. So by Green's theorem,

$$\begin{aligned} W &= \oint_C \mathbf{F}\cdot d\mathbf{r} = \iint_R (2xy)_x - (ye^x - \sin x)_y\; dA = \int_0^1\int_x^1 2y - e^x\; dy\; dx \\ &= \int_0^1 \Big[y^2 - ye^x\Big]_x^1\; dx = \int_0^1 1 - e^x - x^2 + xe^x\; dx \\ &= \int_0^1 1 - x^2 + (x-1)e^x\; dx = \begin{vmatrix} u = x-1 & du = dx \\ dv = e^x\; dx & v = e^x \end{vmatrix} \\ &= \left[x - \frac{x^3}{3} + (x-1)e^x\right]_0^1 - \int_0^1 e^x\; dx = 1 - \frac{1}{3} + 0 - (0-0-1) - \Big[e^x\Big]_0^1 \\ &= \frac{5}{3} - e + 1 = \boxed{\frac{8}{3} - e} \end{aligned}$$

9. Let $f(x,y) = (x-1)^2 + 2y^2$.

(a) Use the appropriate chain rule (not direct substitution) to find $\dfrac{\partial f}{\partial s}$ for $(s,t) = (2,-1)$ if $x = 2st$, $y = t^2 - s$.

Solution: For $(s,t) = (2,-1)$ then $(x,y) = (2(2)(-1), (-1)^2 - 2) = (-4,-1)$. Then,

$$\begin{aligned} \frac{\partial f}{\partial s} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} = 2(x-1)(2t) + 4y(-1) \\ &= 2(-5)(-2) + 4(-1)(-1) = 20 + 4 = \boxed{24}\,. \end{aligned}$$

(b) Use the gradient and Lagrange multipliers to find the absolute minimum and maximum of the function $f(x,y) = (x-1)^2 + 2y^2$ in the region $x^2 + y^2 \leq 4$.

Solution: Extreme values will happen either at critical points within the region or on the boundary.

- critical point(s): $\nabla f(x,y) = \langle 2(x-1), 4y\rangle = \langle 0, 0\rangle$ at $(1,0)$ which is indeed in the region (since $1^2 + 0^2 \leq 4$).

- on the boundary $g(x,y)=x^2+y^2=4$:

$$\nabla f=\lambda\nabla g\ ,\ g(x,y)=4 \quad\Rightarrow\quad \langle 2(x-1),4y\rangle=\lambda\langle 2x,2y\rangle\ ,\ x^2+y^2=4 \quad\Rightarrow\quad \begin{cases}2(x-1)=2\lambda x\\ 4y=2\lambda y\\ x^2+y^2=4\end{cases}$$

From the second equation, we have two cases:
– if $y=0$ then from the constraint: $x^2=4$ so $x=\pm 2$ and so we have the points $(\pm 2,0)$;
– if $y\neq 0$ then $\lambda=2$ and plugging into the first equation we have:

$$2x-2=4x \quad\Rightarrow\quad x=-1$$

which when plugged into the constraint gives you $y^2=3$ so $y=\pm\sqrt{3}$ and so we have the points $(-1,\pm\sqrt{3})$.

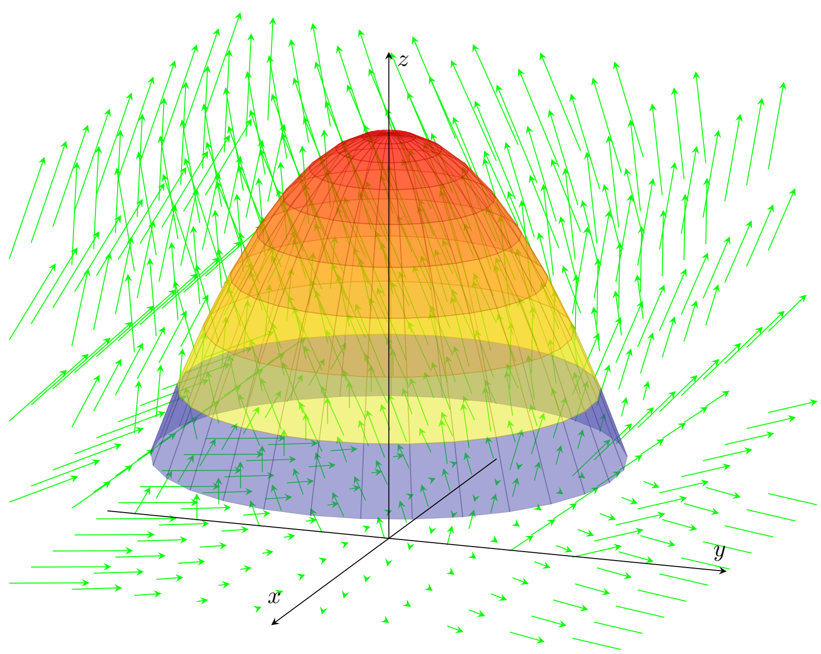
We now compute $f(x,y)$ at all the points found above to find the extreme values:

x	y	$f(x,y)$	
1	0	0	absolute minimum
2	0	1	
-2	0	9	
-1	$\pm\sqrt{3}$	10	absolute maximum

10. Consider the surface S parametrized by

$$\mathbf{r}(u,v)=\left\langle u\cos v, u\sin v, 5-u^2\right\rangle \quad , \quad 0\le u\le 2\ ,\ 0\le v\le 2\pi$$

and a vector field $\mathbf{F}=\left\langle y, y^2-z, 3z\right\rangle$.



(a) Fully set up in (u,v) the flux of the curl across the surface oriented *upwards*. DO NOT evaluate.

$$\iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{N}\,dS = \int_0^{2\pi}\int_0^2 2u^2\cos v-u\,du\,dv$$

Solution: First we compute:

$$\mathbf{r}_u \times \mathbf{r}_v = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \cos v & \sin v & -2u \\ -u\sin v & u\cos v & 0 \end{vmatrix} = \left\langle 2u^2\cos v, 2u^2\sin v, u\right\rangle,$$

and since the $\mathbf{k}$ component is nonnegative, we have the right orientation for $\mathbf{N}$.
Next, we take the curl of the field:

$$\operatorname{curl}\mathbf{F}(x,y,z) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \partial_x & \partial_y & \partial_z \\ y & y^2 - z & 3z \end{vmatrix} = \langle 0+1, 0-0, 0-1\rangle = \langle 1, 0, -1\rangle,$$

and since it is constant we also have $\operatorname{curl}\mathbf{F}(\mathbf{r}(u,v)) = \langle 1,0,-1\rangle$. And therefore,

$$\iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{N}\,dS = \int_0^{2\pi}\int_0^2 \langle 1,0,-1\rangle\cdot\left\langle 2u^2\cos v, 2u^2\sin v, u\right\rangle\,du\,dv = \int_0^{2\pi}\int_0^2 2u^2\cos v - u\,du\,dv$$

(b) Stokes' theorem states that:

$$\iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{N}\,dS = \int_C \mathbf{F}\cdot d\mathbf{r}$$

for C the boundary curve of the surface S oriented here counterclockwise. Give a parametrization in t of C then use it to compute the line integral equivalent to the flux of the curl.

Solution: The boundary curve happens at $u=2$ and so taking $v=t$, a parametrization of C is:

$$\boxed{\mathbf{r}(t) = \langle 2\cos t, 2\sin t, 1\rangle \quad , \quad 0\le t\le 2\pi}$$

where we verify that this goes counterclockwise. Then,

$$d\mathbf{r} = \langle -2\sin t, 2\cos t, 0\rangle\;dt.$$

And so by Stokes' theorem,

$$\begin{aligned}
\iint_S \operatorname{curl}\mathbf{F}\cdot\mathbf{N}\,dS &= \int_0^{2\pi}\mathbf{F}(\mathbf{r}(t))\cdot d\mathbf{r} \\
&= \int_0^{2\pi}\left\langle 2\sin t, (2\sin t)^2 - 1, 3(1)\right\rangle\cdot\langle -2\sin t, 2\cos t, 0\rangle\;dt \\
&= \int_0^{2\pi} -4\sin^2 t + 8\sin^2 t\cos t - 2\cos t\,dt \\
&= \int_0^{2\pi} -2(1-\cos 2t) + 8\sin^2 t\cos t - 2\cos t\,dt \\
&= \left[-2\left(t - \frac{\sin 2t}{2}\right) + \frac{8}{3}\sin^3 t - 2\sin t\right]_0^{2\pi} \\
&= -2(2\pi - 0) + 0 - 0 - (0+0-0) = \boxed{-4\pi}
\end{aligned}$$

(c) Close the surface S by including the portion of the plane $z=1$ that is on the bottom of S. Now use the divergence theorem (stated below) to compute the flux of the vector field across the new closed surface S' as a triple integral (use cylindrical coordinates). *Hint*: The original surface S satisfies $z = 5 - x^2 - y^2$.

$$\oiint_{S'} \mathbf{F}\cdot\mathbf{N}\,dS = \iiint_Q \operatorname{div}\mathbf{F}\,dV$$

Solution: The divergence is:

$$\operatorname{div}\mathbf{F}(x,y,z) = P_x + Q_y + R_z = 0 + 2y + 3.$$

Now if we rewrite our solid Q in cylindrical coordinates, we have $1 \le z \le 5 - r^2$, and so by the divergence theorem,

$$\begin{aligned}
\iint\!\!\!\!\!\!\!\!\bigcirc_{S'} \mathbf{F}\cdot\mathbf{N}\,dS = \iiint_Q \operatorname{div}\mathbf{F}\,dV &= \int_0^{2\pi}\int_0^2\int_1^{5-r^2} (2r\sin\theta + 3)\ r\ dz\ dr\ d\theta \\
&= \int_0^{2\pi}\int_0^2 (2r^2\sin\theta + 3r)\Big[z\Big]_1^{5-r^2}\ dr\ d\theta \\
&= \int_0^{2\pi}\int_0^2 (2r^2\sin\theta + 3r)(4 - r^2)\ dr\ d\theta \\
&= \int_0^{2\pi}\int_0^2 8r^2\sin\theta - 2r^4\sin\theta + 12r - 3r^3\ dr\ d\theta \\
&= \int_0^{2\pi}\left[\frac{8r^3}{3}\sin\theta - \frac{2r^5}{5}\sin\theta + 6r^2 - \frac{3r^4}{4}\right]_0^2 d\theta \\
&= \int_0^{2\pi} \frac{64}{3}\sin\theta - \frac{64}{5}\sin\theta + 24 - 12 - 0\ d\theta \\
&= \left[-\frac{64}{3}\cos\theta + \frac{64}{5}\cos\theta + 12\theta\right]_0^{2\pi} = \boxed{24\pi}\,.
\end{aligned}$$