MATH 371
Midterm Exam 2

Name : Solutions

Nov. 24, 2015

Instructions:

— Show all your work.

— You need not explain your reasoning unless explicitly asked, but doing so increase your chances for partial credit.

— Unless otherwise directed, answers may be in a form where only basic arithmetic (+, −, ×, /, !, exponentials, but not $\binom{n}{m}$) is needed to simplify. If you choose to simplify (to see if an answer is reasonable) mistakes will not be counted against you as long as the correct unsimplified answer is clearly stated.

1. (16 pts. – 4 pts. each) The length of time (in hours) a student takes to work on a one hour exam is a random variable T, modeled by probability density

$$f(t) = \begin{cases} c(1-e^{-t}) & 0 \le t \le 1 \\ 0 & \text{otherwise} \end{cases}$$

where c is some constant.

(a) Determine the value of c.

$$\int_0^1 c(1-e^{-t})\,dt = c(t+e^{-t})\Big|_0^1 = c(1+e^{-1}-1) = ce^{-1}$$

For $ce^{-1}=1$ we need $c=e$

(b) Give the cumulative distribution function for T. (Be sure you indicate for which values of t each part of your formula applies.)

$$F(t) = \int_0^t e(1-e^{-u})\,du = e(u+e^{-u})\Big|_0^t = e(t+e^{-t}-1) \quad \text{when } 0\le t\le 1$$

$$F(t) = \begin{cases} 0 & \text{if } t<0 \\ e(t+e^{-t}-1) & \text{if } 0\le t\le 1 \\ 1 & \text{if } t>1 \end{cases}$$

(c) What is the probability that a student turned in the exam with less than 15 minutes to spare?

$$P(.75\le T\le 1) = F(1)-F(.75) = 1-F(.75)$$
$$= 1-e(3/4+e^{-3/4}-1) = 1+\frac{e}{4}-e^{\frac{1}{4}}$$
$$\approx .395545$$

(d) What is the average time for a student to complete the exam? You may leave your answer as an integral, provided all that remains is to evaluate it.

$$\int_0^1 te(1-e^{-t})\,dt = 2-e/2 \approx .640859 \text{ hrs}$$
$$\approx 38.45 \text{ minutes}$$

2. (12 pts. — 6 pts. each) The probability distribution for two random variables X, Y is specified by the table

		Y			
		-1	0	1	
X	0	.2	.1	.1	.4
	1	.3	.1	.2	.6
		.5	.2	.3	

(a) Are X and Y independent? Justify your answer.

They are <u>not</u> independent

$P(X=0, Y=0) = .1$ but $P(X=0)P(Y=0) = (.4)(.2) = .08$

(b) Find $\text{Cov}(X,Y)$.

$$\text{Cov}(X,Y) = \mathbb{E}(XY) - \mathbb{E}(X)\,\mathbb{E}(Y)$$

$$\mathbb{E}(X) = 0(.4) + 1(.6) = .6$$

$$\mathbb{E}(Y) = -1(.5) + 0(.2) + 1(.3) = -.2$$

$$\mathbb{E}(XY) = \sum_{x=0}^{1} \sum_{y=-1}^{1} xy\,p_{xy} = (1)(-1)(.3) + (1)(1)(.2) = -.1$$

so $\text{Cov}(X,Y) = -.1 - (.6)(-.2) = -.1 + 1.2 = 0.2$

3. The life span of a large number of 60 watt incandescent light bulbs under certain laboratory conditions is recorded, and found to have mean 6 years, with standard deviation 2. Unfortunately, the full data set has been lost, so you cannot create a histogram.

(a) (10 pts.) You want to model the distribution of life spans using one of the following: Uniform, Normal, Exponential, Gamma, or Beta distributions. Using your knowledge of lightbulbs and the qualitative features of these distributions, which one is the most plausible candidates? Explain briefly, giving some rationale for rejecting or retaining each of the distributions as a model.

Lightbulbs have a <u>positive</u> lifespan, with no clear upper limit. This eliminates the Normal, Uniform, and Beta distributions

Since the exponential has mean = standard deviation, we rule it out as well.

A typical shape for the Gamma is  which also seems quite reasonable, so Gamma is the best choice.

(b) (5 pts.) Using the values of the mean and standard deviation that are given, for the distribution you specified in part (a) decide what parameter values you would use to model the distribution of lightbulb life spans.

For Gamma, $\mu = \alpha\beta$ $\quad \text{Var} = \sigma^2 = \alpha\beta^2$

so $6 = \alpha\beta, \quad (2)^2 = \alpha\beta^2 \implies \beta = \frac{2}{3}, \alpha = 9$

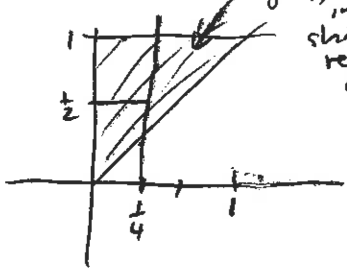
4. (12 pts. — 6 pts. each) The joint density function for random variables X and Y is

$$f(x,y) = \begin{cases} 3y & 0 \le x \le y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) What is $P(Y \le 1/2 \mid X \le 1/4)$? Your answer may be expressed in terms of integrals, provided all that remains is to evaluate them.

$$P(Y \le \tfrac{1}{2} \mid X \le \tfrac{1}{4}) = \frac{P(Y \le \frac{1}{2} \text{ and } X \le \frac{1}{4})}{P(X \le \frac{1}{4})} = \frac{\int_0^{1/4}\int_x^{1/2} 3y \, dy\, dx}{\int_0^{1/4}\int_x^{1} 3y\, dy\, dx}$$

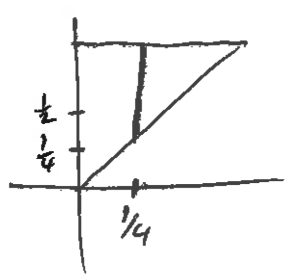
$f(x,y) \ne 0$ in shaded region



$$= \rightsquigarrow = \frac{11}{47}$$

(b) What is $P(Y \le 1/2 \mid X = 1/4)$? Your answer may be expressed in terms of integrals, provided all that remains is to evaluate them.

$$P(Y \le \tfrac{1}{2} \mid X = \tfrac{1}{4}) = \frac{\int_{1/4}^{1/2} 3y\, dy}{\int_{1/4}^{1} 3y\, dy} = \rightsquigarrow = \frac{1}{5}$$



5. A large lot of manufactured items contains some flawed ones. 10% of the lot is defective but repairable, and 5% are so defective that they must be discarded and replaced. Suppose 10 items are selected from the lot, with X being the number that are good, Y being the number that are defective but repairable, and Z being the number that must be replaced. The cost of bringing the selected items up to working order is $C = 2Y + 5Z$.

(a) (4 pts.) What is the probability that $X = 7, Y = 2, Z = 1$?

Using the multinomial distribution

$$P(X=7, Y=2, Z=1) = \binom{10}{7\;2\;1}(.85)^7(.10)^2(.05)^1 = \frac{10!}{7!\,2!}(.85)^7(.10)^2(.05)^1 \approx .0577\ldots$$

(b) (6 pts.) What is the probability that $Z = 1$?

The marginal of a multinomial is binomial, so

$$P(Z=1) = \binom{10}{1}(.05)^1(.95)^9 = 10(.05)(.95)^9 \approx .315124\ldots$$

(c) (7 pts.) What is the mean and variance of C?

$$\mu_c = \mathbb{E}(C) = 2\mathbb{E}(Y) + 5\mathbb{E}(Z) = 2(10)(.10) + 5(10)(.05) = 4.5$$

$$\begin{aligned} Var(C) = Var(2Y+5Z) &= 2^2 Var(Y) + 5^2 Var(Z) + 2(2)(5)\,Cov(Y,Z) \\ &= 4(10)(.10)(.9) + 25(10)(.05)(.95) + 20(-10)(.10)(.05) \\ &= 14.475 \end{aligned}$$

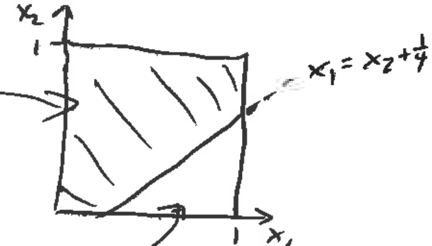
6. (12 pts. — 6 pts. each) Two friends arrange to meet at the Wood Center between noon and 1:00pm. Their arrival times are independent random variables X_1 and X_2 with uniform distributions over the hour. While the first friend is willing to wait as long as necessary to meet his friend, the second is impatient and will wait 15 minutes at most.

(a) Express the event "The friends meet" as an inequality involving X_1 and X_2.

$$X_1 \le X_2 + \frac{1}{4}$$

(b) Calculate the probability that the friends meet.

$$\begin{aligned} P\left(X_1 \le X_2 + \tfrac{1}{4}\right) &= \text{area of} \\ &= 1 - \text{area of} \\ &= 1 - \tfrac{1}{2}\left(\tfrac{3}{4}\right)^2 \\ &= \frac{23}{32} \approx .71875 \end{aligned}$$



7. (16 pts. — 4 pts. each) A certain electronic component in a device has a lifespan given by a random variable $X = X_1$ which is exponentially distributed with mean β. In order to make the device more reliable, an engineer can follow one of two strategies:

(i) Install a back-up component that begins its lifespan when the first one dies. The combined lifespan is then $Y = X_1 + X_2$, where X_2 has the same distribution as X_1.

(ii) Install a more expensive version of the component, whose mean lifetime Z is also exponentially distributed, but with larger mean.

(a) What is the mean of Y? $\mu_Y = \mu_{X_1} + \mu_{X_2} = \beta + \beta = 2\beta$

(b) What is the variance of Y? What assumption did you need to make to answer this question, and why is it reasonable?

$$Var(Y) = Var(X_1) + Var(X_2) + 2\,Cov(X_1, X_2)$$

It's reasonable to assume X_1, X_2 are independent since the lifespan of the first component should not affect that of the 2nd. Then $Cov(X_1, X_2) = 0$

and $Var(Y) = \beta^2 + \beta^2 = 2\beta^2$

(c) Assuming the more expensive component in (ii) is chosen so Z has the same mean as Y, what is the variance of Z?

If $\mu_Z = 2\beta$ then $Var(Z) = (2\beta)^2 = 4\beta^2$ since Z is exponentially distributed.

(d) What does your answer to part (c) tell you about the implications of following these two strategies? Describe informally the differing effects they would have.

While both strategies give the same average lifespan, (ii) leads to more variability — there will be more devices that die much below the average, as well as live beyond the average. In other words, the lifespan of a particular device will be less predictable.