MATH 371
Midterm Exam 1

Name : Solutions

October 13, 2015

Instructions:

— Show all your work.

— You need not explain your reasoning unless explicitly asked, but doing so increase your chances for partial credit.

— Unless otherwise directed, answers may be in a form where only basic arithmetic $(+, -, \times, /, !$, but not $\binom{n}{m}$) is needed to simplify. If you choose to simplify (to see if an answer is reasonable) mistakes will not be counted against you as long as the correct unsimplified answer is clearly stated.

1. (8 pts. – 4 pts. each) You ask a friend to water a plant for you while you are on vacation. Without water, it will die with probability 0.8; with water, it will die with probability 0.15. You are 90% sure your friend will water the plant.

   (a) What is the probability the plant will be alive when you return?

   $P(D|\bar{W}) = .8$
   $P(D|W) = .15$
   $P(W) = .9$

   $$P(D) = P(D|\bar{W})P(\bar{W}) + P(D|W)P(W) = (.8)(.1) + (.15)(.9) = .08 + .135 = .215$$

   $$P(\bar{D}) = 1 - P(D) = 1 - .215 = .785$$

   (b) If it is dead, what is the probability your friend forgot to water it?

   $$P(\bar{W}|D) = \frac{P(\bar{W} \cap D)}{P(D)} = \frac{P(D|\bar{W})P(\bar{W})}{P(D)} = \frac{(.8)(.1)}{.215} = \frac{.08}{.215} \approx .372\ldots$$

2. (10 pts. – 5 pts. each) Five cards are dealt from a well-shuffled deck of playing cards.

   (a) What is the probability that no face cards (J,Q,K) are dealt?

   There are 12 face cards, so $52-12=40$ non-face cards.

   $$\frac{\binom{40}{5}}{\binom{52}{5}} = \frac{\frac{40!}{5!\,35!}}{\frac{52!}{5!\,47!}} = \frac{40!\,47!}{35!\,52!} = \frac{40\cdot39\cdot38\cdot37\cdot36}{52\cdot51\cdot50\cdot49\cdot48} \approx .253$$

   (b) What is the probability that they are all of the same suit (i.e., are a flush)? (Face cards are allowed here.)

   choose a suit

   choose 5 of 13 cards under

   $$\frac{\binom{4}{1}\binom{13}{5}}{\binom{52}{5}} = \frac{4\cdot\frac{13!}{\cancel{5!}\,8!}}{\frac{52!}{\cancel{5!}\,47!}} = \frac{4\cdot13!\,47!}{8!\,52!} = \frac{4\cdot13\cdot12\cdot11\cdot10\cdot9}{52\cdot51\cdot50\cdot49\cdot48}$$

   $\approx .00198$

3. (10 pts. – 5 pts. each)

(a) Suppose $0 < q < 1$. Starting from the formula for the sum of the geometric series, derive a formula for the sum of $\sum_{y=0}^{\infty} yq^{y-1}$.

$$\sum_{y=0}^{\infty} q^y = \frac{1}{1-q}$$

Differentiating with respect to q gives

$$\sum_{y=0}^{\infty} yq^{y-1} = \frac{1}{(1-q)^2}$$

(b) Express the mean of a geometric random variable Y with $p = P(\text{success})$ as a series, and then use your answer to part (a) to express it in "closed form" (i.e., with no infinite summation).

$$\mathbb{E}(Y) = \sum_{y=1}^{\infty} y\,p(Y=y) = \sum_{y=1}^{\infty} yq^{y-1}p = p\sum_{y=1}^{\infty} yq^{y-1} = p\sum_{y=0}^{\infty} yq^{y-1}$$

$$= p\left(\frac{1}{(1-q)^2}\right) = p\left(\frac{1}{p^2}\right) = \frac{1}{p}$$

4. (8 pts. – 4 pts. each) Suppose Y is a random variable with moment generating function $m(t) = e^{.5(e^t-1)}$.

(a) Use the moment generating function to determine the expected value of Y. (Note: you may be able to check your answer if you know what distribution has this mgf. However, the question asks you to *use* the mgf to find $E(Y)$, and you must do that to get credit.)

$$\mathbb{E}(Y) = m'(0) = \left(\frac{d}{dt} e^{.5(e^t-1)}\right)\Big|_{t=0} = e^{.5(e^t-1)}(.5e^t)\Big|_{t=0}$$

$$= e^{.5(1-1)}(.5)\cdot 1 = \underline{.5}$$

(b) Use the moment generating function to determine the variance of Y.

$$\mathbb{E}(Y^2) = m''(0) = \left(\frac{d}{dt} e^{.5(e^t-1)}(.5e^t)\right)\Big|_{t=0} = e^{.5(e^t-1)}(.5e^t)^2 + e^{.5(e^t-1)}(.5e^t)\Big|_{t=0}$$

$$= (.5)^2 + .5$$

$$\text{Var}(Y) = \mathbb{E}(Y^2) - \mathbb{E}(Y)^2 = ((.5)^2 + .5) - (.5)^2 = \underline{.5}$$

5. (12 pts.) A contractor purchases an order of N transistors. To check the quality of the order, he tests 10 of them, and keeps the order only if at least 9 of those pass the test.

(a) (5 pts.) If $N = 20$ and 15% of them are actually defective, what is the probability the order will be kept?

$20\cdot(.15) = 3$ are deffective , 17 are good

$$P(\text{9 or 10 pass test}) = P(\text{9 pass}) + P(\text{10 pass}) = \frac{\binom{17}{9}\binom{3}{1} + \binom{17}{10}\binom{3}{0}}{\binom{20}{10}}$$

$$= \left(3\frac{17!}{9!\,8!} + \frac{17!}{10!\,7!}\right) \Big/ \frac{20!}{10!\,10!} \approx .5$$

(b) (5 pts.) If N is very large and 15% of them are actually defective, what is the probability the order will be kept?

$$P(\text{9 or 10 pass}) = \binom{10}{9}(.85)^9(.15) + \binom{10}{10}(.85)^{10} = 10\,(.85)^9(.15) + (.85)^{10}$$

$$= .544\ldots$$

(c) (2 pts.) What is the name of the distribution you used in part (a)? in part (b)?

a) hypergeometric

b) binomial

6. (10 pts. – 5 pts. each) Exposing an organism to radiation causes mutations in its DNA, and the count of these mutations is usually modeled by the Poisson distribution.

(a) Suppose that on average 2.7 mutations are produced when a certain radiation dosage is applied to a DNA sequence of length 10000 base pairs. However, for a certain study a scientist can use only those sequences with exactly 1 mutation. What faction of the sequences she irradiates will be useful to her?

$$P(X=1) = \frac{\lambda^1}{1!}e^{-\lambda} = 2.7\,e^{-2.7} \approx .1815\ldots$$

(b) Why is the Poisson distribution a reasonable one to describe mutation counts? What are its underlying assumptions, and why are they plausible here?

The Poisson distribution models counts of rare events that are independent. Since mutations here are caused by radiation hitting the DNA, and not many mutations occur, both assumptions are reasonable: A mutation doesn't affect the chance of another mutation, and multiple mutations at one site are negligible.

7. (20 pts. – 4 pts. each) Consider he following simple gambling game: You spin a wheel three times. For each spin there is a probability of $p = 0.7$ of landing on red, and $q = 0.3$ on blue. For each spin that comes up red, you earn \$2 and for each spin that comes up blue, you lose \$5.

(a) Let X be the random variable giving the number of reds in 3 spins. Give its probability distribution in a table. What is $\mathbb{E}(X)$?

$X \sim \text{binomial}(3, .7)$

x	0	1	2	3
$p(x)$	$(.3)^3$	$3(.3)^2(.7)$	$3(.3)(.7)^2$	$(.7)^3$

$\mathbb{E}(X) = np = (3)(.7) = 2.1$

(b) Let Y be the random variable giving your earnings in this game. Give a formula for Y in terms of X.

$Y = 2X - 5(3-X) = 7X - 15$

(c) Using your answer to part (b), find $\mathbb{E}(Y)$. Explain what this tells you about whether you should play this game.

$$\mathbb{E}(Y) = \mathbb{E}(7X-15) = 7\mathbb{E}(X) - 15$$
$$= 7(2.1) - 15$$
$$= 14.7 - 15 = -.30$$

Should not play. On average you will lose \$.30 per game.

(d) Find the variance of your earnings in this game, Var(Y).

$$\text{Var}(Y) = \text{Var}(7X-15) = 7^2\,\text{Var}(X) = 7^2 npq = (49)(3)(.7)(.3)$$
$$= 30.87$$

(e) What should the probabilities p, q be to make the game "fair" (i.e., so $\mathbb{E}(Y) = 0$)?

Since $\mathbb{E}(Y) = 7\mathbb{E}(X) - 15$, we need $\mathbb{E}(X) = \frac{15}{7}$

But $\mathbb{E}(X) = np = 3p$

$3p = \frac{15}{7} \Rightarrow p = \frac{5}{7}, q = \frac{2}{7}$

8. (12 pts.) To conduct a survey on behaviors of those using illegal drugs, you must first find people willing to admit to drug use. Suppose that when asked privately if they use illegal drugs, 80% of the population answers "no". Let X be the number of people you need to survey until you get a single affirmative reply.

(a) (5 pts.) Give a name and formula for the distribution of X, specifying any parameter values. Indicate the sample space for X.

Geometric distribution $P(X=x) = q^{x-1}p$ with $q=.8$, $p=.2$

Sample space for X is $\{1,2,3,\ldots\}$

(b) (5 pts.) What is the probability that you will get an affirmative reply by questioning 10 or fewer people privately? (For full credit, express your answer simply, and *not* as a sum of many terms.)

$$P(X\le 10) = P(X=1) + P(X=2) + \cdots + P(X=10)$$
$$= p + qp + q^2p + \cdots + q^9 p = p(1+q+\cdots+q^9)$$
$$= p\,\frac{1-q^{10}}{1-q} = (.2)\,\frac{1-(.8)^{10}}{1-.8} = 1-(.8)^{10} \approx .8926\ldots$$

(c) (2 pts.) If you question people in small groups rather than privately, would the same distribution apply (possibly with a different parameter value)? Explain briefly.

No. With public questioning, answers are unlikely to be independent. Hearing a "yes" may encourage more "yes"s, and hearing a "no" may discourage "yes"s.

9. (10 pts. – 5 pts. each) Suppose two fair dice are thrown and consider the following events:

A: the first die shows a 4
B: the sum of the two dice is 6
C: the sum of the two dice is 7

(a) Are A and B independent? Prove your answer.

$P(A) = \frac{1}{6}$, $P(B) = \frac{5}{36}$, $P(A\cap B) = \frac{1}{36}$

so $P(A)P(B) \neq P(A\cap B)$

so not independent.

(b) Are A and C independent? Prove your answer.

$P(A) = \frac{1}{6}$, $P(C) = \frac{1}{6}$, $P(A\cap C) = \frac{1}{36}$

so $P(A)P(C) = P(A\cap C)$ so independent.