Math 314
Final Exam

Name: Solutions

May 8, 2013

Show all your work.

1. (10 pts.) Let $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & -2 \\ 1 & 3 & 0 \end{pmatrix}$ and $\mathrm{b} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix}$.

   (a) Find all solutions to $A\mathbf{x} = \mathbf{b}$.

$$\left(\begin{array}{ccc|c} 0 & 1 & 1 & 2 \\ 1 & 1 & -2 & -1 \\ 1 & 3 & 0 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 1 & 3 & 0 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 2 & 2 & 4 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & 1 & -2 & -1 \\ 0 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$$x + y - 2z = -1$$
$$y + z = 2$$
$$z \text{ free}$$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -3+3z \\ 2-z \\ z \end{pmatrix} = \boxed{\begin{pmatrix} -3 \\ 2 \\ 0 \end{pmatrix} + z \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}}$$

$$y = 2 - z$$
$$x = -y + 2z - 1$$
$$= -(2-z) + 2z - 1 = -3 + 3z$$

Use your answer to (a) to answer the following, without additional calculation:

   (b) Give a basis for the nullspace of A. $\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$ (from the solutions in (a))

   (c) Give a basis for the columnspace of A. $\begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \\ 3 \end{pmatrix}$ (the pivot columns of A)

2. (6 pts.)

   (a) (2 pts.) Give a symmetric matrix A so that $\mathbf{x}^T A \mathbf{x} = 2x^2 - 2xy + y^2$

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 1 \end{pmatrix}$$

   (b) (4 pts.) Determine whether A is positive definite. (This can be done several different ways; any correct method is acceptable as long as your work is shown.)

The determinants to check are $|2| = 2$ and $\begin{vmatrix} 2 & -1 \\ -1 & 1 \end{vmatrix} = 2 - 1 = 1$. Both are positive, so A is positive definite

3. (8 pts.) The vectors $\mathbf{v}_1 = (1,0,0,0), \mathbf{v}_2 = (-2,1,0,2), \mathbf{v}_3 = (-1,3,0,1)$ are a basis for a subspace of $\mathbb{R}^4$. Find an orthonormal basis for that subspace.

$w_1 = v_1 = (1,0,0,0)$

$w_2 = v_2 - \text{proj}_{w_1} v_2 = (-2,1,0,2) - \frac{w_1 \cdot v_2}{w_1 \cdot w_1} w_1 = (-2,1,0,2) - \frac{-2}{1}(1,0,0,0) = (0,1,0,2)$

$w_3 = v_3 - \text{proj}_{w_1} v_3 - \text{proj}_{w_2} v_3 = (-1,3,0,1) - \frac{-1}{1}(1,0,0,0) - \frac{5}{5}(0,1,0,2) = (0,2,0,-1)$

$q_1 = \frac{w_1}{\|w_1\|} = (1,0,0,0)$

$q_2 = \frac{w_2}{\|w_2\|} = \left(0, \frac{1}{\sqrt{5}}, 0, \frac{2}{\sqrt{5}}\right)$

$q_3 = \frac{w_3}{\|w_3\|} = \left(0, \frac{2}{\sqrt{5}}, 0, -\frac{1}{\sqrt{5}}\right)$

4. (10 pts.) Find inverses of the following matrices, if they exist. If no inverse exists, explain how you know that.

(a) (3 pts.) $\begin{pmatrix} 1 & 4 \\ -1 & -2 \end{pmatrix}$

$$\frac{1}{-2+4}\begin{pmatrix} -2 & -4 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} -1 & -2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

(b) (7 pts.) $\begin{pmatrix} 1 & 1 & 2 \\ -1 & -1 & -3 \\ 1 & 2 & 1 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ -1 & -1 & -3 & 0 & 1 & 0 \\ 1 & 2 & 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & -1 & 1 & 1 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \\ 0 & 0 & -1 & 1 & 1 & 0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1 & 1 & 2 & 1 & 0 & 0 \\ 0 & 1 & -1 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 1 & 0 & 3 & 2 & 0 \\ 0 & 1 & 0 & -2 & -1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 5 & 3 & -1 \\ 0 & 1 & 0 & -2 & -1 & 1 \\ 0 & 0 & 1 & -1 & -1 & 0 \end{array}\right)$$

$$\begin{pmatrix} 5 & 3 & -1 \\ -2 & -1 & 1 \\ -1 & -1 & 0 \end{pmatrix}$$

5. (10 pts.) The annual changes in the population of an organism is modeled by $\mathbf{p}_{t+1} = A\mathbf{p}_t$, where $A = \begin{pmatrix} 0 & 3 \\ 0.2 & 0.7 \end{pmatrix}$. The first entry of $\mathbf{p}_t$ refers to young, and the second to adults.

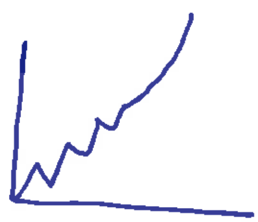
(a) (2 pts.) What is the biological meaning of the 0.2 in this matrix?

20% of the young become adults each year

(b) (2 pts.) The diagonalization $A = S\Lambda S^{-1}$ has $\Lambda = \begin{pmatrix} 1.2 & 0 \\ 0 & -.5 \end{pmatrix}$ and $S = \begin{pmatrix} 5 & 6 \\ 2 & -1 \end{pmatrix}$. What is the stable age distribution of this model? What does it tell you about the population?

$\begin{pmatrix} 5 \\ 2 \end{pmatrix}$. In the long term, the population will have roughly 5 young for every 2 adults.

(c) (3 pts.) Sketch a graph of time vs. population size for the two groups in this model that indicates typical qualitative behavior you should see if the initial population, $\mathbf{p}_0$, is randomly chosen.



exponential growth (due to $\lambda = 1.2$) with overlay of decaying oscillations (due to $\lambda = -.5$)

(d) (3 pts.) If the initial population were $\mathbf{p}_0 = (10, 4)$ (twice the first column of S), what would $\mathbf{p}_{20}$ be?

$(1.2)^{20} \begin{pmatrix} 10 \\ 4 \end{pmatrix}$

6. (8 pts. — 4 pts. each) Decide whether each of these transformations $T : \mathbb{R}^2 \to R^2$ is linear or not. If it is not linear, explain why it is not. If it is linear, give a matrix to express it (using the standard bases).

(a) $T(x, y) = (x, y) + (1, -1)$.

Not linear. $T(0,0) \neq \vec{0}$

(b) $T(x, y) = (y, 0)$.

Linear. $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$

7. (8 pts. — 4 pts. each) Let $A = \begin{pmatrix} 3 & 1 & 4 \\ 0 & 1 & 5 \\ 0 & 1 & 5 \end{pmatrix}$.

(a) Find all eigenvalues of A.

$$\begin{vmatrix} 3-\lambda & 1 & 4 \\ 0 & 1-\lambda & 5 \\ 0 & 1 & 5-\lambda \end{vmatrix} = (3-\lambda)\left[(1-\lambda)(5-\lambda)-5\right] = (3-\lambda)(\lambda^2-6\lambda) = (3-\lambda)\lambda(\lambda-6)$$

$$\lambda = 0, 3, 6$$

(b) Find an eigenvector for the *largest* eigenvalue of A.

$\lambda = 6$

$$A - 6I = \begin{pmatrix} -3 & 1 & 4 \\ 0 & -5 & 5 \\ 0 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} -3 & 1 & 4 \\ 0 & -5 & 5 \\ 0 & 0 & 0 \end{pmatrix}$$

$-3x + y + 4z = 0$
$-5y + 5z = 0$
z free

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \frac{5}{3}z \\ z \\ z \end{pmatrix} = z\begin{pmatrix} 5/3 \\ 1 \\ 1 \end{pmatrix} \qquad \vec{v} = \begin{pmatrix} 5 \\ 3 \\ 3 \end{pmatrix}$$

8. (8 pts.) Let $A = \begin{pmatrix} 1 & 1 & 2 & 3 \\ 2 & -1 & 1 & 0 \\ 1 & 0 & 1 & 1 \end{pmatrix}$.

(a) (6 pts.) Let V be the column space of A. Find a basis for $V^\perp$.

$V = \mathcal{C}(A) \qquad V^\perp = \mathcal{N}(A^T)$

$$\begin{pmatrix} 1 & 2 & 1 \\ 1 & -1 & 0 \\ 2 & 1 & 1 \\ 3 & 0 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & -3 & -1 \\ 0 & -3 & -1 \\ 0 & -6 & -2 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & 1 \\ 0 & -3 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$x + 2y + z = 0$
$-3y - z = 0$
z free

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -\frac{1}{3}z \\ -\frac{1}{3}z \\ z \end{pmatrix} = z\begin{pmatrix} -1/3 \\ -1/3 \\ 1 \end{pmatrix} \qquad \begin{pmatrix} -1 \\ -1 \\ 3 \end{pmatrix}$$

(b) (2 pts.) What is the rank of A? (If you did part (a) you should be able to answer this with no further computation.)

$\text{rank}(A) = 2$

9. (8 pts.) Four data points in the plane are $(x, y) = (-2, 2)$, $(-1, 2)$, $(0, 1)$, $(1, -1)$.

(a) (2 pts.) Give a matrix equation that you would *like to solve* (but which has no solution) to find the equation of a line $y = mx + b$ through these points.

$$\begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 1 \\ -1 \end{pmatrix}$$

(b) (6 pts.) Give a matrix equation that *can be solved* to find the least squares best-fit line for these points. (Do not solve it.)

$$\begin{pmatrix} -2 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -2 & 1 \\ -1 & 1 \\ 0 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -2 & -1 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 2 \\ 1 \\ -1 \end{pmatrix}$$

$$\begin{pmatrix} 6 & -2 \\ -2 & 4 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -7 \\ 4 \end{pmatrix}$$

10. (24 pts. — 2 pts. each) Give short answers.

(a) We can be sure a real square matrix has real eigenvalues if it is symmetric

(b) If for some specific $A, \mathbf{b}$, where A is 4×5 and $\mathbf{b} \in \mathbb{R}^4$, we know $A\mathbf{x} = \mathbf{b}$ has no solution, then the rank of A must be (list all possibilities) 0, 1, 2, 3

(c) If for some specific $A, \mathbf{b}$, where A is 4×5 and $\mathbf{b} \in \mathbb{R}^4$, we know $A\mathbf{x} = \mathbf{b}$ has a 2-dimensional plane of solutions, then the rank of A must be (list all possibilities) 3

(d) If for some specific A, where A is 4×5 we know $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{b}$, then the rank of A must be (list all possibilities) 4

(e) If Q is an orthogonal matrix, then its determinant can only be ± 1.
(Hint: What is Q^TQ?)

(f) Using the 'big formula' to find the determinant of a 7×7 matrix would requiring adding (or subtracting) $7!$ terms, each of which is the product of 7 numbers. For such a matrix, it would be much easier to compute the determinant by elimination.

(g) Using Cramer's rule to solve $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} e \\ f \end{pmatrix}$ gives $x =$ $\dfrac{\begin{vmatrix} e & b \\ f & d \end{vmatrix}}{\begin{vmatrix} a & b \\ c & d \end{vmatrix}} = \dfrac{ed-bf}{ad-bc}$

(h) A 4×4 matrix with eigenvalues 1,3,3,3 could have as few as 2 eigenvectors. But it must have 4 orthogonal eigenvectors if A is symmetric

(i) If $\mathbf{a}, \mathbf{b}, \mathbf{c}$ in $\mathbb{R}^3$ form three edges of a parallelopiped, then its volume can be computed as $\text{abs}\left(\det\begin{pmatrix} - v_1 - \\ - v_2 - \\ - v_3 - \end{pmatrix}\right)$

(j) A matrix that rotates vectors in $\mathbb{R}^2$ about the origin by an angle of θ counterclockwise is $R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$

(k) If the SVD of a matrix A is

$$U\Sigma V^T = \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 10 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{pmatrix}$$

then the rank of A is 1 and the pseudoinverse A^+ is (you may leave your answer as a product):

$$V\begin{pmatrix} 1/10 & 0 \\ 0 & 0 \end{pmatrix} U^T$$

(l) If A has eigenvector $\mathbf{v}$ with eigenvalue λ, then MAM^{-1} will have eigenvector $M\vec{v}$ with eigenvalue λ.