Math 314 Name : Solutions
Final Exam May 7, 2008

Show all your work. If you are told to use a particular method, you can get full credit for the problem ONLY if you use the specified method; other methods will receive partial or no credit.

1. Let $A = \begin{pmatrix} 2 & 5 \\ 4 & 3 \end{pmatrix}$.

(a) (9 pts.) Give an invertible matrix $S$ and a diagonal matrix $\Lambda$ such that $A = S\Lambda S^{-1}$.

$$|A - \lambda I| = (2-\lambda)(3-\lambda) - 20 = 0$$
$$\lambda^2 - 5\lambda - 14 = 0$$
$$(\lambda - 7)(\lambda + 2) = 0$$

$$\Lambda = \begin{pmatrix} 7 & 0 \\ 0 & -2 \end{pmatrix}$$

$\underline{\lambda = 7}$

$$A - 7I = \begin{pmatrix} -5 & 5 \\ 4 & -4 \end{pmatrix} \to \begin{pmatrix} -5 & 5 \\ 0 & 0 \end{pmatrix} \quad -5x + 5y = 0 \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} \to \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$\underline{\lambda = -2}$

$$A + 2I = \begin{pmatrix} 4 & 5 \\ 4 & 5 \end{pmatrix} \to \begin{pmatrix} 4 & 5 \\ 0 & 0 \end{pmatrix} \quad 4x + 5y = 0 \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5/4\, y \\ y \end{pmatrix} = y\begin{pmatrix} -5/4 \\ 1 \end{pmatrix}$$

$$S = \begin{pmatrix} 1 & -5 \\ 1 & 4 \end{pmatrix}$$

(b) (3 pts.) Give invertible matrices $T$ and $U$ and diagonal matrices $L$ and $M$ such that $A^{100} = TLT^{-1}$ and $A^{-1} = UMU^{-1}$.

$$T = U = S$$

$$L = \begin{pmatrix} 7^{100} & 0 \\ 0 & -2^{100} \end{pmatrix} \qquad M = \begin{pmatrix} \frac{1}{7} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

2. (8 pts.) Use Gaussian elimination to calculate the inverse of $A = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 2 & 6 \\ 0 & -2 & 0 \end{pmatrix}$, if it exists.

$$\left(\begin{array}{ccc|ccc} 0 & 2 & 3 & 1 & 0 & 0 \\ 2 & 2 & 6 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 2 & 2 & 6 & 0 & 1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 2 & 2 & 6 & 0 & 1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 1 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 2 & 2 & 0 & -2 & 1 & -2 \\ 0 & 2 & 0 & 0 & 0 & -1 \\ 0 & 0 & 3 & 1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 2 & 0 & 0 & -2 & 1 & -1 \\ 0 & 2 & 0 & 0 & 0 & -1 \\ 0 & 0 & 3 & 1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & \frac{1}{2} & -1/2 \\ 0 & 1 & 0 & 0 & 0 & -1/2 \\ 0 & 0 & 1 & 1/3 & 0 & 1/3 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -1 & \frac{1}{2} & -1/2 \\ 0 & 0 & -1/2 \\ 1/3 & 0 & 1/3 \end{pmatrix}$$

3. Let $A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & -2 & 4 \\ 1 & -2 & 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 6 \\ 4 \end{pmatrix}$.

(a) (8 pts.) Find all solutions to $A\mathbf{x} = \mathbf{b}$.

$$\left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ -1 & 0 & 1 & 2 \\ 0 & -2 & 4 & 6 \\ 1 & -2 & 3 & 4 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & -1 & 2 & 3 \\ 0 & -2 & 4 & 6 \\ 0 & -1 & 2 & 3 \end{array}\right) \to \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & -1 & 2 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right)$$

$z$ free

$x - y + z = 1$

$-y + 2z = 3$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -2+z \\ -3+2z \\ z \end{pmatrix} = \begin{pmatrix} -2 \\ -3 \\ 0 \end{pmatrix} + z\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

(b) (4 pts.) Give an $LU$ factorization of $A$. (Your work in part (a) should make this easy.)

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -1 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

(c) (2 pts.) Give a basis for the nullspace of $A$.

$$\begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$$

(d) (2 pts.) Give a basis for the columnspace of $A$.

$$\begin{pmatrix} 1 \\ -1 \\ 0 \\ 1 \end{pmatrix}, \begin{pmatrix} -1 \\ 0 \\ -2 \\ -2 \end{pmatrix}$$

4. (9 pts. – 3 pts. each) Give definitions of the italicized terms:

(a) the *dimension* of a vector space is the number of vectors in any basis

(b) the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are *linearly independent* if $c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_k\vec{v}_k = \vec{0}$ only when all $c_i = 0$

(c) a *basis* for a vector space is a set of vectors that is linearly independent and spans the space.

5. Let $\mathbf{v}_1 = (1, 2, 0, -1)$, $\mathbf{v}_2 = (2, 2, -1, 0)$ and $\mathbf{v}_3 = (0, 1, -4, 2)$ and $V = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(a) (6 pts.) Find an orthogonal basis for $V$. (You may find an orthonormal basis if you wish, but you need not.)

$$w_1 = v_1 = (1,2,0,-1)$$

$$w_2 = v_2 - \text{proj}_{w_1} v_2 = v_2 - \frac{w_1 \cdot v_2}{w_1 \cdot w_1} w_1 = (2,2,-1,0) - \frac{6}{6}(1,2,0,-1) = (1,0,-1,1)$$

$$w_3 = v_3 - \text{proj}_{w_1} v_3 - \text{proj}_{w_2} v_3$$

$$= (0,1,-4,2) - \frac{w_1 \cdot v_3}{w_1 \cdot w_1} w_1 - \frac{w_2 \cdot v_3}{w_2 \cdot w_2} w_2$$

$$= (0,1,-4,2) - 0w_1 - \frac{6}{3}(1,0,-1,1) = (-2,1,-2,0)$$

(b) (3 pts.) Find the projection of $(1, 1, 1, 1)$ onto $V$. (You may leave your answer as a linear combination of vectors, without simplifying.)

$$\text{proj}_V(1,1,1,1) = \text{proj}_{w_1}(1,1,1,1) + \text{proj}_{w_2}(1,1,1,1) + \text{proj}_{w_3}(1,1,1,1)$$

$$= \frac{2}{6}(1,2,0,-1) + \frac{1}{3}(1,0,-1,1) + \frac{1}{9}(-2,1,-2,0)$$

6. (8 pts. – 2 pts. each) Suppose you are given $b$ vectors in $\mathbb{R}^a$ that span a space of dimension $c$.

(a) Say as much as you can about the relationships between $a$, $b$ and $c$. (For example, an incorrect answer might be $a = b \leq c$)

$b \geq c$, $a \geq c$

(b) What is the dimension of the orthogonal complement of the span of the vectors?

$a - c$

(c) If the given vectors are independent, what more can you say about $a$, $b$, and $c$?

$b = c \leq a$

(d) If the given vectors span $\mathbb{R}^a$ (but are not necessarily independent), what more can you say about $a$, $b$, and $c$?

$b \geq c = a$

7. (6 pts. – 2 pts. each) We'd like to find the equation of a straight line $y = mx + b$ through the data points $(-1, 2)$, $(0, 1)$, and $(2, -3)$. Unfortunately, these points are not on a line.

(a) In matrix form, write a system of equations (that has no solution) that you'd like to solve to find $m$ and $b$.

$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$$

(b) Give a system of equations that you could solve to find the least-squares best-fit line for the three data points.

$$\begin{pmatrix} -1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -1 & 0 & 2 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} 2 \\ 1 \\ -3 \end{pmatrix}$$

$$\begin{pmatrix} 5 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -8 \\ 0 \end{pmatrix}$$

(c) Use the formula for the inverse of a $2 \times 2$ matrix to find the solution of the system in part (b). If you didn't get part (b), solve $\begin{pmatrix} 5 & 1 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$ instead.

$$\begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 5 & 1 \\ 1 & 3 \end{pmatrix}^{-1}\begin{pmatrix} -8 \\ 0 \end{pmatrix} = \frac{1}{14}\begin{pmatrix} 3 & -1 \\ -1 & 5 \end{pmatrix}\begin{pmatrix} -8 \\ 0 \end{pmatrix} = \frac{1}{14}\begin{pmatrix} -24 \\ 8 \end{pmatrix} = \begin{pmatrix} -8/7 \\ 4/7 \end{pmatrix}$$

8. (10 pts. – 5 pts. each) The set of all $2 \times 2$ matrices with real number entries forms a vector space $M$.

(a) Consider the set of all $2 \times 2$ matrices which have a 1 in the upper right corner. Is this a subspace of $M$? Justify your answer.

No, $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix} = \begin{pmatrix} 3 & 2 \\ 4 & 5 \end{pmatrix}$ does not have a 1 in the upper right

(or $5\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 5 & 5 \\ 5 & 5 \end{pmatrix}$ does not have a 1 in the upper right)

(b) Consider the set of all $2 \times 2$ matrices which have a 0 in the upper right corner. Is this a subspace of $M$? Justify your answer.

Yes $\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} + \begin{pmatrix} d & 0 \\ e & f \end{pmatrix} = \begin{pmatrix} a+d & 0 \\ b+e & c+f \end{pmatrix}$

$$d\begin{pmatrix} a & 0 \\ b & c \end{pmatrix} = \begin{pmatrix} ad & 0 \\ bd & cd \end{pmatrix}$$

9. In $\mathbb{R}^4$, let $W$ be the set of vectors $(x, y, z, w)$ that satisfy the equations

$$x - 3y + 2z + w = 0,$$
$$x - 3y + z + 2w = 0$$

(a) (3 pts.) Explain why $W$ is a subspace of $\mathbb{R}^4$.

$W = \mathcal{N}(A)$ for $A = \begin{pmatrix} 1 & -3 & 2 & 1 \\ 1 & -3 & 1 & 2 \end{pmatrix}$

(b) (3 pts.) Find a basis for $W$.

$\begin{pmatrix} 1 & -3 & 2 & 1 \\ 1 & -3 & 1 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & 2 & 1 \\ 0 & 0 & -1 & 1 \end{pmatrix}$ $y, w$ free

$x - 3y + 2z + w = 0$

$-z + w = 0$

$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 3y - 3w \\ y \\ w \\ w \end{pmatrix} = y \begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix} + w \begin{pmatrix} -3 \\ 0 \\ 1 \\ 1 \end{pmatrix}$ basis is $\begin{pmatrix} 3 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} -3 \\ 0 \\ 1 \\ 1 \end{pmatrix}$

(c) (2 pts.) Find a basis for $W^{\perp}$.

i.e. basis for rowspace of $A$

$\begin{pmatrix} 1 \\ -3 \\ 2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \\ -1 \\ 1 \end{pmatrix}$

(d) (1 pt.) What is the dimension of $W$? 2

10. (14 pts. – 2 pts. each) Complete the following.

(a) If an $n \times n$ matrix has a non-zero determinant, then its columnspace is ...

$\mathbb{R}^n$

(b) The main conceptual idea behind least-squares solutions to a system $A\mathbf{x} = \mathbf{b}$ is that if there is no solution to the original system we should... project $b$ onto $\mathcal{C}(A)$

(c) In order for a matrix equation $A\mathbf{x} = \mathbf{b}$ (where $A$ is $m \times n$) to have a solution regardless of what $\mathbf{b}$ is, we need the rank of $A$ to be $\underline{\ m\ }$, so the columnspace of $A$ is $\underline{\ \mathbb{R}^m\ }$.

(d) In order for a matrix equation $A\mathbf{x} = \mathbf{b}$ (where $A$ is $m \times n$) to have at most one solution we need the rank of $A$ to be $\underline{\ n\ }$, so the nullspace of $A$ is $\underline{\ \{\vec{0}\}\ }$.

(e) In order to solve $A\mathbf{x} = \mathbf{b}$, it is generally a really stupid idea to find $A^{-1}$ since ... this is more work than solving $Ax=b$ by Gaussian elimination.

(f) The determinant of a square matrix is related to its eigenvalues by ...

$\det A = \lambda_1 \lambda_2 \cdots \lambda_n$

(g) The best way to calculate a determinant for a large matrix is ... by Gaussian elimination