

Show all your work. If you are told to use a particular method, you can get full credit for the problem ONLY if you use the specified method; other methods will receive partial or no credit.

1. Let  $A = \begin{pmatrix} 2 & 5 \\ 4 & 3 \end{pmatrix}$ .

(a) (9 pts.) Give an invertible matrix  $S$  and a diagonal matrix  $\Lambda$  such that  $A = S\Lambda S^{-1}$ .

$$|A - \lambda I| = (2 - \lambda)(3 - \lambda) - 20 = 0$$

$$\lambda^2 - 5\lambda - 14 = 0$$

$$(\lambda - 7)(\lambda + 2) = 0$$

$$\Lambda = \begin{pmatrix} 7 & 0 \\ 0 & -2 \end{pmatrix}$$

$$\lambda = 7$$

$$A - 7I = \begin{pmatrix} -5 & 5 \\ 4 & -4 \end{pmatrix} \rightarrow \begin{pmatrix} -5 & 5 \\ 0 & 0 \end{pmatrix} \quad -5x + 5y = 0 \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ y \end{pmatrix} \rightarrow \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

$$\lambda = -2$$

$$A + 2I = \begin{pmatrix} 4 & 5 \\ 4 & 5 \end{pmatrix} \rightarrow \begin{pmatrix} 4 & 5 \\ 0 & 0 \end{pmatrix} \quad 4x + 5y = 0 \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -5/4 y \\ y \end{pmatrix} \rightarrow \begin{pmatrix} -5/4 \\ 1 \end{pmatrix}$$

$$S = \begin{pmatrix} 1 & -5 \\ 1 & 4 \end{pmatrix}$$

(b) (3 pts.) Give invertible matrices  $T$  and  $U$  and diagonal matrices  $L$  and  $M$  such that  $A^{100} = TLT^{-1}$  and  $A^{-1} = UMU^{-1}$ .

$$T = U = S$$

$$L = \begin{pmatrix} 7^{100} & 0 \\ 0 & -2^{100} \end{pmatrix} \quad M = \begin{pmatrix} \frac{1}{7} & 0 \\ 0 & -\frac{1}{2} \end{pmatrix}$$

2. (8 pts.) Use Gaussian elimination to calculate the inverse of  $A = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 2 & 6 \\ 0 & -2 & 0 \end{pmatrix}$ , if it exists.

$$\left( \begin{array}{ccc|ccc} 0 & 2 & 3 & 1 & 0 & 0 \\ 2 & 2 & 6 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 \end{array} \right) \rightarrow \left( \begin{array}{ccc|ccc} 2 & 2 & 6 & 0 & 1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 1 \end{array} \right) \rightarrow \left( \begin{array}{ccc|ccc} 2 & 2 & 6 & 0 & 1 & 0 \\ 0 & 2 & 3 & 1 & 0 & 0 \\ 0 & 0 & 3 & 1 & 0 & 1 \end{array} \right)$$

$$\rightarrow \left( \begin{array}{ccc|ccc} 2 & 2 & 0 & -2 & 1 & -2 \\ 0 & 2 & 0 & 0 & 0 & -1 \\ 0 & 0 & 3 & 1 & 0 & 1 \end{array} \right) \rightarrow \left( \begin{array}{ccc|ccc} 2 & 0 & 0 & -2 & 1 & -1 \\ 0 & 2 & 0 & 0 & 0 & -1 \\ 0 & 0 & 3 & 1 & 0 & 1 \end{array} \right) \rightarrow \left( \begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & 0 & 0 & 0 & -\frac{1}{2} \\ 0 & 0 & 1 & \frac{1}{3} & 0 & \frac{1}{3} \end{array} \right)$$

$$A^{-1} = \begin{pmatrix} -1 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \\ \frac{1}{3} & 0 & \frac{1}{3} \end{pmatrix}$$

