Math 314 Name :__________________

Final Exam May 7, 2008

Show all your work. If you are told to use a particular method, you can get full credit for the problem ONLY if you use the specified method; other methods will receive partial or no credit.

1. Let $A = \begin{pmatrix} 2 & 5 \\ 4 & 3 \end{pmatrix}$.

   (a) (9 pts.) Give an invertible matrix S and a diagonal matrix Λ such that $A = S\Lambda S^{-1}$.

   (b) (3 pts.) Give invertible matrices T and U and diagonal matrices L and M such that $A^{100} = TLT^{-1}$ and $A^{-1} = UMU^{-1}$.

2. (8 pts.) Use Gaussian elimination to calculate the inverse of $A = \begin{pmatrix} 0 & 2 & 3 \\ 2 & 2 & 6 \\ 0 & -2 & 0 \end{pmatrix}$, if it exists.

3. Let $A = \begin{pmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 0 & -2 & 4 \\ 1 & -2 & 3 \end{pmatrix}$, $\mathbf{b} = \begin{pmatrix} 1 \\ 2 \\ 6 \\ 4 \end{pmatrix}$.

(a) (8 pts.) Find all solutions to $A\mathbf{x} = \mathbf{b}$.

(b) (4 pts.) Give an LU factorization of A. (Your work in part (a) should make this easy.)

(c) (2 pts.) Give a basis for the nullspace of A.

(d) (2 pts.) Give a basis for the columnspace of A.

4. (9 pts. – 3 pts. each) Give definitions of the italicized terms:

(a) the *dimension* of a vector space

(b) the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are *linearly independent*

(c) a *basis* for a vector space

5. Let $\mathbf{v}_1 = (1, 2, 0, -1)$, $\mathbf{v}_2 = (2, 2, -1, 0)$ and $\mathbf{v}_3 = (0, 1, -4, 2)$ and $V = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$.

(a) (6 pts.) Find an orthogonal basis for V. (You may find an orthonormal basis if you wish, but you need not.)

(b) (3 pts.) Find the projection of $(1, 1, 1, 1)$ onto V. (You may leave your answer as a linear combination of vectors, without simplifying.)

6. (8 pts. – 2 pts. each) Suppose you are given b vectors in $\mathbb{R}^a$ that span a space of dimension c.

(a) Say as much as you can about the relationships between a, b and c. (For example, an incorrect answer might be $a = b \leq c$)

(b) What is the dimension of the orthogonal complement of the span of the vectors?

(c) If the given vectors are independent, what more can you say about a, b, and c?

(d) If the given vectors span $\mathbb{R}^a$ (but are not necessarily independent), what more can you say about a, b, and c?

7. (6 pts. – 2 pts. each) We'd like to find the equation of a straight line $y = mx + b$ through the data points $(-1, 2)$, $(0, 1)$, and $(2, -3)$. Unfortunately, these points are not on a line.

(a) In matrix form, write a system of equations (that has no solution) that you'd like to solve to find m and b.

(b) Give a system of equations that you could solve to find the least-squares best-fit line for the three data points.

(c) Use the formula for the inverse of a 2×2 matrix to find the solution of the system in part (b). If you didn't get part (b), solve $\begin{pmatrix} 5 & 1 \\ 1 & 3 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 7 \\ 3 \end{pmatrix}$ instead.

8. (10 pts. – 5 pts. each) The set of all 2×2 matrices with real number entries forms a vector space M.

(a) Consider the set of all 2×2 matrices which have a 1 in the upper right corner. Is this a subspace of M? Justify your answer.

(b) Consider the set of all 2×2 matrices which have a 0 in the upper right corner. Is this a subspace of M? Justify your answer.

9. In $\mathbb{R}^4$, let W be the set of vectors (x, y, z, w) that satisfy the equations

$$x - 3y + 2z + w = 0,$$
$$x - 3y + z + 2w = 0$$

(a) (3 pts.) Explain why W is a subspace of $\mathbb{R}^4$.

(b) (3 pts.) Find a basis for W.

(c) (2 pts.) Find a basis for $W^{\perp}$.

(d) (1 pt.) What is the dimension of W?

10. (14 pts. – 2 pts. each) Complete the following.

(a) If an $n \times n$ matrix has a non-zero determinant, then its columnspace is . . .

(b) The main conceptual idea behind least-squares solutions to a system $A\mathbf{x} = \mathbf{b}$ is that if there is no solution to the original system we should...

(c) In order for a matrix equation $A\mathbf{x} = \mathbf{b}$ (where A is $m \times n$) to have a solution regardless of what $\mathbf{b}$ is, we need the rank of A to be ______, so the columnspace of A is ______.

(d) In order for a matrix equation $A\mathbf{x} = \mathbf{b}$ (where A is $m \times n$) to have at most one solution we need the rank of A to be ______, so the nullspace of A is ______.

(e) In order to solve $A\mathbf{x} = \mathbf{b}$, it is generally a really stupid idea to find A^{-1} since ...

(f) The determinant of a square matrix is related to its eigenvalues by ...

(g) The best way to calculate a determinant for a large matrix is ...