Math 314
Exam 2

Name : Solutions

November 21, 2014

Show your work on all problems.

1. (15 pts.) A subspace V of $\mathbb{R}^4$ has basis $\mathbf{v}_1 = (1, 0, -1, 1), \mathbf{v}_2 = (0, 1, -2, 1), \mathbf{v}_3 = (-1, 0, 0, 1)$. Find an orthonormal basis for V.

$$\vec{w}_1 = \vec{v}_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}$$

$$\vec{w}_2 = \vec{v}_2 - \text{proj}_{\vec{w}_1}\vec{v}_2 = \begin{pmatrix} 0 \\ 1 \\ -2 \\ 1 \end{pmatrix} - \frac{(0\ 1\ -2\ 1)\begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}}{(1\ 0\ -1\ 1)\begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \\ -2 \\ 1 \end{pmatrix} - \frac{3}{3}\begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\vec{w}_3 = \vec{v}_3 - \text{proj}_{\vec{w}_1}\vec{v}_3 - \text{proj}_{\vec{w}_2}\vec{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \frac{(-1\ 0\ 0\ 1)\begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}}{(1\ 0\ -1\ 1)\begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ 0 \\ -1 \\ 1 \end{pmatrix} - \frac{(-1\ 0\ 0\ 1)\begin{pmatrix} -1 \\ 1 \\ -1 \\ 0 \end{pmatrix}}{(-1\ 1\ -1\ 0)\begin{pmatrix} -1 \\ 1 \\ -1 \\ 0 \end{pmatrix}} \begin{pmatrix} -1 \\ 1 \\ -1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 0 \\ 0 \\ 1 \end{pmatrix} - \vec{0} - \frac{1}{3}\begin{pmatrix} -1 \\ 1 \\ -1 \\ 0 \end{pmatrix} = \begin{pmatrix} -2/3 \\ -1/3 \\ 1/3 \\ 1 \end{pmatrix}, \text{ or } \begin{pmatrix} -2 \\ -1 \\ 1 \\ 3 \end{pmatrix}$$

$$\vec{q}_1 = \frac{1}{\|\vec{w}_1\|}\vec{w}_1 = \begin{pmatrix} 1/\sqrt{3} \\ 0 \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{pmatrix} \quad \vec{q}_2 = \frac{1}{\|\vec{w}_2\|}\vec{w}_2 = \begin{pmatrix} -1/\sqrt{3} \\ 1/\sqrt{3} \\ -1/\sqrt{3} \\ 0 \end{pmatrix} \quad \vec{q}_3 = \frac{1}{\|\vec{w}_3\|}\vec{w}_3 = \begin{pmatrix} -2/\sqrt{15} \\ -1/\sqrt{15} \\ 1/\sqrt{15} \\ 3/\sqrt{15} \end{pmatrix}$$

2. (12 pts.–4 pts. each) Four data points from an experiment are

x	-1	0	1	2
y	2	1	3	7

(a) Give a matrix equation that you would *like* to solve (but that probably doesn't have a solution) to find a quadratic of the form $y = ax^2 + bx + c$ that passes through the 4 data points.

$$\begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ 3 \\ 7 \end{pmatrix}$$

(b) Give a matrix equation that *could be* solved to find the least-squares best-fit quadratic for (a). (Do not simplify or solve.)

$$\begin{pmatrix} 1 & 0 & 1 & 4 \\ -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{pmatrix} \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 & 4 \\ -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ 3 \\ 7 \end{pmatrix}$$

(c) Briefly explain the key idea behind passing from your equation in (a) to that in (b). (Any good answer will use the word "projection").

Since $A\vec{x} = \vec{b}$ is not solvable, we replace $\vec{b}$ with the closest vector to it in $C(A)$, so $A\vec{x} = \text{proj}_{C(A)}\vec{b}$ is solvable. Then $A\vec{x} = A(A^TA)^{-1}A^T\vec{b}$ can be simplified by multiplying by A^T to get $A^TA\vec{x} = A^T\vec{b}$

3. (12 pts.) In $\mathbb{R}^4$, a subspace W is spanned by $(1,0,1,0)$ and $(0,2,1,1)$. Find a basis for $W^\perp$.

Let $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{pmatrix}$ so $W = \mathcal{C}(A^T)$. Then $W^\perp = \mathcal{N}(A)$.

No elimination steps are needed on A to find $\mathcal{N}(A)$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 1 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -z \\ -\frac{1}{2}z - \frac{1}{2}w \\ z \\ w \end{pmatrix} = z\begin{pmatrix} -1 \\ -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix} + w\begin{pmatrix} 0 \\ -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$$

Basis for $W^\perp$ is $\begin{pmatrix} -1 \\ -\frac{1}{2} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -\frac{1}{2} \\ 0 \\ 1 \end{pmatrix}$

4. (14 pts.) Let $A = \begin{pmatrix} 0 & 1 & -2 \\ 1 & -1 & 0 \\ -2 & 1 & -1 \end{pmatrix}$.

(a) (10 pts.) Compute $|A|$ *by elimination.* (No other methods will receive credit).

$$A \xrightarrow[\text{exchg}]{\text{row}} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ -2 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & -3 \end{pmatrix}$$

$$|A| = (-1)(1)(1)(-3) = 3$$

(b) (4 pts.) Give the upper right (row 1, column 3) entry of A^{-1}.

$$A^{-1} = \frac{1}{|A|} C^T \quad \text{so } (A^{-1})_{13} = \frac{1}{3} C_{31} = \frac{1}{3}\left(+\begin{vmatrix} 1 & -2 \\ -1 & 0 \end{vmatrix}\right) = \frac{-2}{3}$$

5. (12 pts.) A matrix $A = \begin{pmatrix} .8 & .2 \\ .3 & .7 \end{pmatrix}$ has eigenvectors $(1,1)$ and $(2,-3)$, with respective eigenvalues 1 and 0.5.

(a) (5 pts.) Give a diagonalization $A = S\Lambda S^{-1}$ of the matrix.

$$A = \begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & .5 \end{pmatrix}\frac{1}{-5}\begin{pmatrix} -3 & -2 \\ -1 & 1 \end{pmatrix}$$

(b) (7 pts.) Use your diagonalization to compute $\lim_{k\to\infty} A^k$.

$$\lim_{k\to\infty} A^k = \lim_{k\to\infty} S\Lambda^k S^{-1} = S \lim_{k\to\infty}\begin{pmatrix} 1^k & 0 \\ 0 & .5^k \end{pmatrix} S^{-1} = \begin{pmatrix} 1 & 2 \\ 1 & -3 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\left(-\frac{1}{5}\right)\begin{pmatrix} -3 & -2 \\ -1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}\left(-\frac{1}{5}\right)\begin{pmatrix} -3 & -2 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} 3/5 & 2/5 \\ 3/5 & 2/5 \end{pmatrix}$$

6. (14 pts.–7 pts. each) Let $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

(a) Find the eigenvalues of A.

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0 \iff (2-\lambda)^2 - 1 = 0 \iff \lambda^2 - 4\lambda + 3 = 0$$
$$(\lambda-3)(\lambda-1) = 0$$

$$\lambda = 1, 3$$

(b) Find an eigenvector for the *smallest* eigenvalue of A.

$\underline{\lambda=1}$ $\quad A - 1 \cdot I = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ $\quad N(A-1I)$: $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ $\quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$$\vec{v} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

7. (6 pts.) Give a formula for the value of y in the solution to:

$$\begin{aligned} 2x - y \quad + z &= 1 \\ -x + 2y \qquad &= 2 \\ 3x + y \quad + 2z &= 1 \end{aligned}$$

(Do not simplify your answer.)

$$y = \frac{\begin{vmatrix} 2 & 1 & 1 \\ -1 & 2 & 0 \\ 3 & 1 & 2 \end{vmatrix}}{\begin{vmatrix} 2 & -1 & 1 \\ -1 & 2 & 0 \\ 3 & 1 & 2 \end{vmatrix}}$$

8. (15 pts.–3 pts.each) Complete the following:

(a) If P is a permutation matrix, then $\det P$ is $\underline{\ \pm 1\ }$. (*Give all possibilities.*)

(b) If V is a k-dimensional subspace of $\mathbb{R}^n$, then $V^\perp$ is $\underline{(n-k)}$-dimensional.

(c) If P is a projection matrix then $P^2 = \underline{\ P\ }$.

(d) If a "warped box" (parallelepiped) in 3-d has edges given by the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, then its volume is most easily computed as... $\left|\det\begin{pmatrix} - v_1 - \\ - v_2 - \\ - v_3 - \end{pmatrix}\right|$

(e) If A is 4×4 with $|A| = -3$, then $|-A| = \underline{\ -3\ }$, $|A^T| = \underline{\ -3\ }$, and $|A^{-1}| = \underline{\ -1/3\ }$.

(f) The "big formula" for the determinant of an $n \times n$ matrix is a sum and difference of $\underline{\ n!\ }$ terms, each of which is a product of $\underline{\ n\ }$ entries of the matrix.

(g) If the columns of A are independent, the formula for a matrix that projects onto the column space of A is ...

$$P = A(A^TA)^{-1}A^T$$