

Show all your work.

1. (16 pts.) Let  $A = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$ .

(a) (6 pts.) Find the eigenvalues of  $A$ .

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & -1 \\ -2 & -\lambda \end{vmatrix} = (1-\lambda)(-\lambda) - 2 = \lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1) = 0$$

So  $\lambda = 2, -1$

(b) (6 pts.) For each eigenvalue, determine an eigenvector.

$\lambda = 2$   
 $A - 2I = \begin{pmatrix} -1 & -1 \\ -2 & -2 \end{pmatrix} \rightarrow \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix}$   $-x - y = 0$   
 $y$  free  $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y \begin{pmatrix} -1 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} -1 \\ 1 \end{pmatrix}$

$\lambda = -1$   
 $A + I = \begin{pmatrix} 2 & -1 \\ -2 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix}$   $2x - y = 0$   
 $y$  free  $\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}y \\ y \end{pmatrix} = y \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix}$   
(or  $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ )

(c) (4 pts.) Give matrices  $\Lambda$  and  $S$  for a diagonalization  $A = S\Lambda S^{-1}$ .

$$\Lambda = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \quad S = \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}$$

2. (11 pts.) In  $\mathbb{R}^4$ , a subspace  $V$  has basis  $(1, 2, -1, 1)$ ,  $(2, 0, 1, 1)$ . Find a basis for  $V^\perp$ .

Find all vectors  $\perp$  to  $(1, 2, -1, 1)$  &  $(2, 0, 1, 1)$ , i.e. solve

$$\begin{aligned} x + 2y - z + w &= 0 \\ 2x + z + w &= 0 \end{aligned}$$

i.e. find  $\mathcal{N}\left(\begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 0 & 1 & 1 \end{pmatrix}\right)$  (Note:  $V = \text{row space of } \begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 0 & 1 & 1 \end{pmatrix}$ )  
So  $V^\perp = \text{nullspace of } \begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 0 & 1 & 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 0 & 1 & 1 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & -4 & 3 & -1 \end{pmatrix} \quad \left. \begin{aligned} x + 2y - z + w &= 0 \\ -4y + 3z - w &= 0 \end{aligned} \right\} \begin{aligned} x &= -2y + z - w \\ y &= \frac{3}{4}z - \frac{1}{4}w \end{aligned} \quad \left. \begin{aligned} x &= -2\left(\frac{3}{4}z - \frac{1}{4}w\right) + z - w \\ &= -\frac{1}{2}z - \frac{1}{2}w \end{aligned} \right\}$$

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}z - \frac{1}{2}w \\ \frac{3}{4}z - \frac{1}{4}w \\ z \\ w \end{pmatrix} = z \begin{pmatrix} -\frac{1}{2} \\ \frac{3}{4} \\ 1 \\ 0 \end{pmatrix} + w \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{4} \\ 0 \\ 1 \end{pmatrix} \rightarrow \begin{pmatrix} -\frac{1}{2} \\ \frac{3}{4} \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ -\frac{1}{4} \\ 0 \\ 1 \end{pmatrix}$$

3. (15 pts.) The vectors

$$v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, v_3 = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \end{pmatrix}$$

span a 3-d subspace of  $\mathbb{R}^4$ . Find an orthonormal basis for that subspace.

$$w_1 = v_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

$$w_2 = v_2 - \text{proj}_{w_1} v_2 = v_2 - \frac{w_1 \cdot v_2}{w_1 \cdot w_1} w_1 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} - \frac{2}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}$$

$$w_3 = v_3 - \text{proj}_{w_1} v_3 - \text{proj}_{w_2} v_3 = \begin{pmatrix} 2 \\ 0 \\ 2 \\ 0 \end{pmatrix} - \frac{2}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix} - \frac{2}{2} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

$$\hat{g}_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \hat{g}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \hat{g}_3 = \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \end{pmatrix}$$

4. (15 pts.—5 pts. each) Suppose you wanted to fit a straight line  $y = mx + b$  to the  $(x, y)$  data points

$$(-1, 2), (0, 1), (1, -2), (2, -2).$$

(a) Give, in matrix form, a system of 4 equations in 2 unknowns that you would *like* to solve to find this line, even though this system has no solution.

$$\begin{aligned} m(-1) + b &= 2 \\ m(0) + b &= 1 \\ m(1) + b &= -2 \\ m(2) + b &= -2 \end{aligned}$$

$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \\ -2 \\ -2 \end{pmatrix}$$

(b) Give, in matrix form, a system of 2 equations in 2 unknown that you could solve to find the least-squares best fit line. (Do NOT solve them.)

$$\begin{pmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 1 \\ -2 \\ -2 \end{pmatrix}$$

$$\begin{pmatrix} 6 & 2 \\ 2 & 4 \end{pmatrix} \begin{pmatrix} m \\ b \end{pmatrix} = \begin{pmatrix} -8 \\ -1 \end{pmatrix}$$

(c) The idea behind what is being done here is that if  $Ax = b$  has no solution due to suspected errors in  $b$ , then we should replace  $b$  with a different vector so the system becomes solvable. This vector is found by projecting  $b$  onto what?

Onto the column space of A

