Math 314
Exam 2

Name : Solutions
April 10, 2013

Show all your work.

1. (16 pts.) Let $A = \begin{pmatrix} 1 & -1 \\ -2 & 0 \end{pmatrix}$.

   (a) (6 pts.) Find the eigenvalues of A.

$$|A-\lambda I| = \begin{vmatrix} 1-\lambda & -1 \\ -2 & -\lambda \end{vmatrix} = (1-\lambda)(-\lambda) - 2 = \lambda^2 - \lambda - 2 = (\lambda-2)(\lambda+1) = 0$$

$$\text{So } \lambda = 2, -1$$

   (b) (6 pts.) For each eigenvalue, determine an eigenvector.

$\underline{\lambda=2}$

$$A-2I = \begin{pmatrix} -1 & -1 \\ -2 & -2 \end{pmatrix} \to \begin{pmatrix} -1 & -1 \\ 0 & 0 \end{pmatrix} \quad \begin{matrix} -x-y=0 \\ y \text{ free} \end{matrix} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -y \\ y \end{pmatrix} = y\begin{pmatrix} -1 \\ 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$\underline{\lambda=-1}$

$$A+I = \begin{pmatrix} 2 & -1 \\ -2 & 1 \end{pmatrix} \to \begin{pmatrix} 2 & -1 \\ 0 & 0 \end{pmatrix} \quad \begin{matrix} 2x-y=0 \\ y \text{ free} \end{matrix} \quad \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \frac{1}{2}y \\ y \end{pmatrix} = y\begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \frac{1}{2} \\ 1 \end{pmatrix} \left(\text{or } \begin{pmatrix} 1 \\ 2 \end{pmatrix}\right)$$

   (c) (4 pts.) Give matrices Λ and S for a diagonalization $A = S\Lambda S^{-1}$.

$$\Lambda = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \quad S = \begin{pmatrix} -1 & 1 \\ 1 & 2 \end{pmatrix}$$

2. (11 pts.) In $\mathbb{R}^4$, a subspace V has basis $(1,2,-1,1)$, $(2,0,1,1)$. Find a basis for $V^\perp$.

Find all vectors $\perp$ to $(1,2,-1,1)$ + $(2,0,1,1)$, i.e. solve

$$\begin{aligned} 1x + 2y - z + w &= 0 \\ 2x \quad\quad + z + w &= 0 \end{aligned}$$

i.e. find $\mathcal{N}\left(\begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 0 & 1 & 1 \end{pmatrix}\right)$ (Note: V = row space of $\begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 0 & 1 & 1 \end{pmatrix}$ so $V^\perp$ = nullspace of ")

$$\begin{pmatrix} 1 & 2 & -1 & 1 \\ 2 & 0 & 1 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -1 & 1 \\ 0 & -4 & 3 & -1 \end{pmatrix} \quad \left.\begin{matrix} x+2y-z+w=0 \\ -4y+3z-w=0 \end{matrix}\right\} \begin{matrix} x = -2y+z-w \\ y = \frac{3}{4}z - \frac{1}{4}w \end{matrix} \Big\} \begin{aligned} x &= -2\left(\tfrac{3}{4}z - \tfrac{1}{4}w\right) + z - w \\ &= -\tfrac{1}{2}z - \tfrac{1}{2}w \end{aligned}$$

z, w free

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} -\frac{1}{2}z - \frac{1}{2}w \\ \frac{3}{4}z - \frac{1}{4}w \\ z \\ w \end{pmatrix} = z\begin{pmatrix} -1/2 \\ 3/4 \\ 1 \\ 0 \end{pmatrix} + w\begin{pmatrix} -1/2 \\ -1/4 \\ 0 \\ 1 \end{pmatrix} \rightsquigarrow \begin{pmatrix} -1/2 \\ 3/4 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -1/2 \\ -1/4 \\ 0 \\ 1 \end{pmatrix}$$

3. (15 pts.) The vectors

$$\mathbf{v}_1 = \begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \mathbf{v}_2 = \begin{pmatrix}1\\1\\1\\1\end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix}2\\0\\2\\0\end{pmatrix}$$

span a 3-d subspace of $\mathbb{R}^4$. Find an orthonormal basis for that subspace.

$$w_1 = v_1 = \begin{pmatrix}1\\1\\0\\0\end{pmatrix}$$

$$w_2 = v_2 - \mathrm{proj}_{w_1} v_2 = v_2 - \frac{w_1 \cdot v_2}{w_1 \cdot w_1} w_1 = \begin{pmatrix}1\\1\\1\\1\end{pmatrix} - \frac{2}{2}\begin{pmatrix}1\\1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\0\\1\\1\end{pmatrix}$$

$$w_3 = v_3 - \mathrm{proj}_{w_1} v_3 - \mathrm{proj}_{w_2} v_3 = \begin{pmatrix}2\\0\\2\\0\end{pmatrix} - \frac{2}{2}\begin{pmatrix}1\\1\\0\\0\end{pmatrix} - \frac{2}{2}\begin{pmatrix}0\\0\\1\\1\end{pmatrix} = \begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}$$

$$q_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}1\\1\\0\\0\end{pmatrix}, \quad q_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\0\\1\\1\end{pmatrix}, \quad q_3 = \frac{1}{2}\begin{pmatrix}1\\-1\\1\\-1\end{pmatrix}$$

4. (15 pts.—5 pts. each) Suppose you wanted to fit a straight line $y = mx + b$ to the (x, y) data points

$$(-1, 2),\ (0, 1),\ (1, -2),\ (2, -2).$$

(a) Give, in matrix form, a system of 4 equations in 2 unknowns that you would *like* to solve to find this line, even though this system has no solution.

$$\begin{aligned} m(-1) + b &= 2 \\ m(0) + b &= 1 \\ m(1) + b &= -2 \\ m(2) + b &= -2 \end{aligned} \qquad \begin{pmatrix}-1 & 1\\ 0 & 1\\ 1 & 1\\ 2 & 1\end{pmatrix}\begin{pmatrix}m\\b\end{pmatrix} = \begin{pmatrix}2\\1\\-2\\-2\end{pmatrix}$$

(b) Give, in matrix form, a system of 2 equations in 2 unknown that you could solve to find the least-squares best fit line. (Do NOT solve them.)

$$\begin{pmatrix}-1 & 0 & 1 & 2\\ 1 & 1 & 1 & 1\end{pmatrix}\begin{pmatrix}-1 & 1\\ 0 & 1\\ 1 & 1\\ 2 & 1\end{pmatrix}\begin{pmatrix}m\\b\end{pmatrix} = \begin{pmatrix}-1 & 0 & 1 & 2\\ 1 & 1 & 1 & 1\end{pmatrix}\begin{pmatrix}2\\1\\-2\\-2\end{pmatrix}$$

$$\begin{pmatrix}6 & 2\\ 2 & 4\end{pmatrix}\begin{pmatrix}m\\b\end{pmatrix} = \begin{pmatrix}-8\\-1\end{pmatrix}$$

(c) The idea behind what is being done here is that if $A\mathbf{x} = \mathbf{b}$ has no solution due to suspected errors in $\mathbf{b}$, then we should replace $\mathbf{b}$ with a different vector so the system becomes solvable. This vector is found by projecting $\mathbf{b}$ onto what? Onto the column space of A

5. (12 pts.) Suppose

$$A = \begin{pmatrix} 2 & 1 & 1 & -1 & 2 \\ 4 & 2 & 3 & -1 & 5 \\ 2 & 1 & -2 & -4 & -1 \\ -2 & -1 & 0 & 2 & -1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ 1 & -3 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 1 & -1 & 2 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(a) (3 pts.) Give a basis for the rowspace of A.

$(2,1,1,-1,2), (0,0,1,1,1)$ the pivot rows of $\underline{U}$

(b) (3 pts.) Give a basis for the column space of A.

$(2,4,2,-2), (1,3,-2,0)$. the pivot columns of $\underline{A}$

(c) (3 pts.) What is the dimension of the nullspace of A? 3

(d) (3 pts.) What is the dimension of the left nullspace of A? 2

6. (10 pts.) Calculate the determinant

$$\begin{vmatrix} 1 & 1 & 2 & 1 \\ 1 & 1 & 3 & 3 \\ 2 & 1 & -1 & 1 \\ 1 & 1 & 2 & 4 \end{vmatrix}.$$

$$\begin{vmatrix} 1 & 1 & 2 & 1 \\ 1 & 1 & 3 & 3 \\ 2 & 1 & -1 & 1 \\ 1 & 1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 2 & 1 & -1 & 1 \\ 1 & 1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & -5 & -1 \\ 1 & 1 & 2 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 2 \\ 0 & -1 & -5 & -1 \\ 0 & 0 & 0 & 3 \end{vmatrix} = - \begin{vmatrix} 1 & 1 & 2 & 1 \\ 0 & -1 & -5 & -1 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 3 \end{vmatrix}$$

$$= -(1)(-1)(1)(3) = 3$$

7. (21 pts.—3 pts. each) Fill in the blanks:

(a) If the columns of an $m \times r$ matrix A are independent, then a matrix to project $\mathbb{R}^m$ onto the column space of A can be found using the formula $A(A^TA)^{-1}A^T$

(b) If the 'big formula' for the determinant of a 5×5 matrix were written out, it would be a sum of $5! = 120$ terms, each of which is ± 1 times a product of 5 entries of the matrix.

(c) The inverse of an orthogonal matrix Q is most easily computed by $Q^{-1} = Q^T$

(d) If a 3×3 matrix A has eigenvalues $3, 1, -2$, then the eigenvalues of A^3 are $3^3, 1^3, (-2)^3$, or $27, 1, -8$

(e) If $\det B = -13$, then $\det B^T =$ -13.

(f) If P is a 5×5 matrix that projects vectors in $\mathbb{R}^5$ onto a 3-dimensional subspace W, then the 5 eigenvalues of P will be $1, 1, 1, 0, 0$

(g) If $|A| = 2/5$, then $|A^{-1}| =$ $5/2$.