Math 314
Exam 2

Name : Key
April 4, 2008

1. (15 pts. – 5 pts. each) A matrix A and its reduced row echelon form U are:

$$A = \begin{pmatrix} 1 & 0 & -1 & 1 & 1 \\ 3 & 1 & -1 & 4 & 2 \\ 1 & 1 & 1 & 2 & 0 \\ 2 & 1 & 0 & 3 & 1 \\ 1 & 1 & 1 & 2 & -3 \\ 4 & -3 & -10 & 1 & 7 \end{pmatrix}, \quad U = \begin{pmatrix} 1 & 0 & -1 & 1 & 0 \\ 0 & 1 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

(a) Give a basis for the nullspace of A.

$$A\vec{x}=\vec{0} \iff U\vec{x}=\vec{0} \iff \begin{matrix} x - z + w = 0 \\ y + 2z + w = 0 \\ v = 0 \\ z, w \text{ free} \end{matrix} \iff \begin{matrix} x = z - w \\ y = -2z - w \\ v = 0 \end{matrix} \iff \begin{pmatrix} x \\ y \\ z \\ w \\ v \end{pmatrix} = \begin{pmatrix} z-w \\ -2z-w \\ z \\ w \\ 0 \end{pmatrix} = z\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + w\begin{pmatrix} -1 \\ -1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$$

basis $= \{(1,-2,1,0,0), (-1,-1,0,1,0)\}$

(b) Give a basis for the rowspace of A.

basis $= \{(1,0,-1,1,0), (0,1,2,1,0), (0,0,0,0,1)\}$

(c) Give a basis for the columnspace of A.

basis $= \{(1,3,1,2,1,4), (0,1,1,1,1,-3), (1,2,0,1,-3,7)\}$

2. (6 pts.)What is the rank of the matrix $B = \begin{pmatrix} 1 & -3 & 2 & 3 \\ 2 & 3 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 3 & -6 & 7 & 8 \end{pmatrix}$? Show your work.

$$B \to \begin{pmatrix} 1 & -3 & 2 & 3 \\ 0 & 9 & -3 & -6 \\ 0 & 3 & -1 & -2 \\ 0 & 3 & -1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & 2 & 3 \\ 0 & 9 & -3 & -6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix} \to \begin{pmatrix} 1 & -3 & 2 & 3 \\ 0 & 9 & -3 & -6 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\operatorname{rank}(B) = 3$

3. (12 pts.) Let $A = \begin{pmatrix} 0 & -2 & 1 \\ 1 & 2 & 3 \\ 2 & 2 & 2 \end{pmatrix}$.

(a) (6 pts.) Compute $|A|$ using elimination. (No credit for other methods.)

$$A \xrightarrow[\text{exchange}]{\text{row}} \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & 1 \\ 2 & 2 & 2 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & 1 \\ 0 & -2 & -4 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 2 & 3 \\ 0 & -2 & 1 \\ 0 & 0 & -5 \end{pmatrix}$$

$$\det A = \underset{\substack{\uparrow \\ \text{row exchange}}}{(-1)}(1)(-2)(-5) = -10$$

(b) (2 pts.) What is $|A^{-1}|$?

$$|A^{-1}| = \frac{1}{|A|} = -\frac{1}{10}$$

(c) (2 pts.) What is $|2A|$?

$$|2A| = 2^3|A| = -80$$

(d) (2 pts.) What is $|A^4|$?

$$|A^4| = |A|^4 = 10{,}000$$

4. (9 pts.) Use Cramer's rule to solve the system of equations $\begin{cases} 2x + 3y = 1, \\ 7x + 11y = 3. \end{cases}$ (No credit for other methods.)

$$x = \frac{\begin{vmatrix} 1 & 3 \\ 3 & 11 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 7 & 11 \end{vmatrix}} = \frac{2}{1} = 2 \qquad y = \frac{\begin{vmatrix} 2 & 1 \\ 7 & 3 \end{vmatrix}}{\begin{vmatrix} 2 & 3 \\ 7 & 11 \end{vmatrix}} = \frac{-1}{1} = -1$$

5. (10 pts.) Find the inverse of $C = \begin{pmatrix} -1 & 0 & 2 \\ 0 & 1 & 1 \\ 1 & -2 & 1 \end{pmatrix}$ using the formula involving cofactors. (No credit for other methods.)

$$C^{-1} = \frac{1}{\det C}\left(\text{Cofactor}(C)\right)^T \qquad \text{Cofactor}(C) = \begin{pmatrix} 3 & 1 & -1 \\ -4 & -3 & -2 \\ -2 & 1 & -1 \end{pmatrix}$$

$$\det C = (-1)(3) + 0(1) + 2(-1) = -5$$

$$C^{-1} = \begin{pmatrix} -3/5 & 4/5 & 2/5 \\ -1/5 & 3/5 & -1/5 \\ 1/5 & 2/5 & 1/5 \end{pmatrix}$$

6. (6 pts.) For a computer graphics system, the point (x, y) in the plane $\mathbb{R}^2$ is encoded by $(x, y, 1)$ so that 3×3 matrices can be used to rotate and translate points. Give a 3×3 matrix that will produce the combined action of first translating by 2 units in the y-direction, and then rotating by $90°$ clockwise.

$$\left(\begin{array}{cc|c} 0 & 1 & 0 \\ -1 & 0 & 0 \\ \hline 0 & 0 & 1 \end{array}\right)\left(\begin{array}{cc|c} 1 & 0 & 0 \\ 0 & 1 & 2 \\ \hline 0 & 0 & 1 \end{array}\right) = \begin{pmatrix} 0 & 1 & 2 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

7. (12 pts. – 4 pts. each) For the vector space $\mathbb{P}_5$ of polynomials of degree at most 5, answer the following.

(a) Is the subset of $\mathbb{P}_5$ whose elements are those polynomials with a quadratic term of $2x^2$ a subspace? Explain why or why not.

No. $2x^2$ is in the set but $2(2x^2) = 4x^2$ is not, so the set is not closed under scalar multiplication

-OR- $2x^2$ and $2x^2+1$ are in the set but $(2x^2) + (2x^2+1) = 4x^2+1$ is not, so set is not closed under addition

-OR- the zero polynomial is not in the set

(b) Is the subset of $\mathbb{P}_5$ whose elements are those polynomials with no quadratic term a subspace? Explain why or why not.

Yes. Adding two polynomials with no x^2 term, or multiplying one by a constant, gives a polynomial with no x^2 term. Thus the set is closed under both addition and scalar multiplication.

(c) What is the dimension of $\mathbb{P}_5$? Why?

$\dim(\mathbb{P}_5) = 6$ because $\mathbb{P}_5$ has as a basis $1, x, x^2, \ldots, x^5$ which has 6 elements

8. (12 pts. – 6 pts. each) Consider the bases $\mathcal{B} = \{(1, -2), (3, 1)\}$ and $\mathcal{C} = \{(-2, 1), (3, 7)\}$ of $\mathbb{R}^2$.

(a) If $\mathbf{v} = (-1, -5)$, find the coordinates $[\mathbf{v}]_{\mathcal{B}}$ of $\mathbf{v}$ with respect to $\mathcal{B}$.

Solve $\begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix}[v]_{\mathcal{B}} = \begin{pmatrix} -1 \\ -5 \end{pmatrix}$ by any method (GE, Cramer, finding $\begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix}^{-1}$)

$$[v]_{\mathcal{B}} = \frac{1}{7}\begin{pmatrix} 1 & -3 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 \\ -5 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$

(b) Give a matrix $P_{\mathcal{B}\to\mathcal{C}}$ such that $[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{B}\to\mathcal{C}}[\mathbf{x}]_{\mathcal{B}}$ for all vectors $\mathbf{x} \in \mathbb{R}^2$.

- Either -

$$P_{\mathcal{B}\to\mathcal{C}} = \begin{pmatrix} -2 & 3 \\ 1 & 7 \end{pmatrix}^{-1}\begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix} = -\frac{1}{17}\begin{pmatrix} 7 & -3 \\ -1 & -2 \end{pmatrix}\begin{pmatrix} 1 & 3 \\ -2 & 1 \end{pmatrix} = \begin{pmatrix} -13/17 & -18/17 \\ -3/17 & 5/17 \end{pmatrix}$$

-or-

$$\left(\begin{array}{cc|cc} -2 & 3 & 1 & 3 \\ 1 & 7 & -2 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 7 & -2 & 1 \\ -2 & 3 & 1 & 3 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 7 & -2 & 1 \\ 0 & 17 & -3 & 5 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 7 & -2 & 1 \\ 0 & 1 & -3/17 & 5/17 \end{array}\right)$$

$$\to \left(\begin{array}{cc|cc} 1 & 0 & -13/17 & -18/17 \\ 0 & 1 & -3/17 & 5/17 \end{array}\right) \quad \text{so} \quad P_{\mathcal{B}\to\mathcal{C}} = \begin{pmatrix} -13/17 & -18/17 \\ -3/17 & 5/17 \end{pmatrix}$$

9. (18 pts. – 6 pts. each) Give short answers, with explanations.

   (a) An 18×23 matrix A has a 7-dimensional nullspace. Will $A\mathbf{x} = \mathbf{b}$ be solvable for every $\mathbf{b} \in \mathbb{R}^{18}$? Briefly explain your reasoning.

   No. $\dim Nul(A) = 23 - \text{rank}(A)$ so $\text{rank}(A) = 16$. Thus $\dim Col(A) = 16$. Therefore $A\vec{x} = \vec{b}$ is only solvable for $\vec{b}$ in a 16-dim subspace of $\mathbb{R}^{18}$

   -OR-
   GE on A must produce 7 free variables, so $23 - 7 = 16$ pivots. Thus it produces 2 rows of 0's, so $A\vec{x} = \vec{b}$ will not be solvable for all $\vec{b}$

   (b) If A is an $m \times n$ matrix and the linear transformation $T_A : \mathbb{R}^{n} \to \mathbb{R}^{m}$ is onto, what can you say about the rank of A? Briefly explain your reasoning.

   $Col(A) = \mathbb{R}^m$, so A has rank m

   -OR-
   $A\vec{x} = \vec{b}$ is solvable for all $\vec{b}$, so GE must produce a pivot in every row, so A has rank m

   (c) Suppose A is a non-zero 3×3 matrix, with rows $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. If $\det A = 0$, then what are the possible dimensions of Span$\{\mathbf{a}, \mathbf{b}, \mathbf{c}\}$? (List all possibilities.) Briefly explain your reasoning.

   $\det A = 0$ tells use the rows are dependent, so their span has dimension < 3. Since A is non-zero, some row is non-zero so the dimension of the span is > 0. So the possible dimensions are 1, 2 and easy examples show both occur (e.g. $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $A = \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$)

   -OR-
   $\det A = 0$ tells us the volume of the parallelopiped determined by $\vec{a}, \vec{b}, \vec{c}$ is 0, so the 3 must lie in a plane. Since at least one is non-zero, the span must be 1 or 2 dimensional