Math 314
Exam 1

Name : Solutions
February 27, 2013

1. (12 pts.) Find the inverse of the following matrix (or show none exists). Show all your work.

$$A = \begin{pmatrix} 1 & 1 & 0 \\ 2 & 2 & 2 \\ -1 & 0 & 2 \end{pmatrix}$$

$$\left(\begin{array}{ccc|ccc} 1&1&0&1&0&0\\ 2&2&2&0&1&0\\ -1&0&2&0&0&0 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1&1&0&1&0&0\\ 0&0&2&-2&1&0\\ 0&1&2&1&0&1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1&1&0&1&0&0\\ 0&1&2&1&0&1\\ 0&0&2&-2&1&0 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1&1&0&1&0&0\\ 0&1&0&3&-1&1\\ 0&0&2&-2&1&0 \end{array}\right)$$

$$\to \left(\begin{array}{ccc|ccc} 1&0&0&-2&1&-1\\ 0&1&0&3&-1&1\\ 0&0&2&-2&1&0 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1&0&0&-2&1&-1\\ 0&1&0&3&-1&1\\ 0&0&1&-1&\frac{1}{2}&0 \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} -2 & 1 & -1 \\ 3 & -1 & 1 \\ -1 & 1/2 & 0 \end{pmatrix}$$

2. (26 pts.) Consider the matrix equation $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{pmatrix} 1 & 1 & -1 & 0 \\ 1 & 1 & -2 & 2 \\ 2 & 2 & -1 & -2 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 2 \\ 1 \\ 5 \end{pmatrix}$$

(a) (13 pts.) Find all solutions, showing your work.

$$\left(\begin{array}{cccc|c} 1&1&-1&0&2\\ 1&1&-2&2&1\\ 2&2&-1&-2&5 \end{array}\right) \to \left(\begin{array}{cccc|c} 1&1&-1&0&2\\ 0&0&-1&2&-1\\ 0&0&1&-2&1 \end{array}\right) \to \left(\begin{array}{cccc|c} 1&1&-1&0&2\\ 0&0&-1&2&-1\\ 0&0&0&0&0 \end{array}\right)$$

$x + y - z = 2$
$-z + 2w = -1$
y, w free, x, z pivot
$z = 1 + 2w$
$x = 2 + z - y = 2 + (1+2w) - y$
$= 3 + 2w - y$

(alternatively, do Jordan elimination:

$$\to \left(\begin{array}{cccc|c} 1&1&0&-2&3\\ 0&0&1&-2&1\\ 0&0&0&0&0 \end{array}\right) \quad \left.\begin{array}{l} x+y-2w=3 \\ z-2w=1 \end{array}\right\} \Rightarrow \begin{array}{l} x = 3 - y + 2w \\ z = 1 + 2w \end{array} \Rightarrow \text{)}$$

So $\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 3+2w-y \\ y \\ 1+2w \\ w \end{pmatrix} = \boxed{\begin{pmatrix} 3 \\ 0 \\ 1 \\ 0 \end{pmatrix} + y\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix} + w\begin{pmatrix} 2 \\ 0 \\ 2 \\ 1 \end{pmatrix}}$

(b) (3 pts.) Do the solutions to this problem form a subspace of $\mathbb{R}^4$? (CIRCLE ONE) Yes / No

$\vec{0}$ is not a solution to $A\vec{x} = \vec{b}$ So the solutions are not a subspace (The only time solutions form a subspace is if $\vec{b} = \vec{0}$)

(c) (6 pts.) Based on your answer to part (a), give a basis for the nullspace of A.

$$\begin{pmatrix} -1 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 2 \\ 0 \\ 2 \\ 1 \end{pmatrix}$$

(d) (4 pts.) If $\mathbf{b}$ were changed to be a different vector in $\mathbb{R}^3$, then $A\mathbf{x} = \mathbf{b}$ will (CIRCLE ONE)
(1) certainly be solvable,
(2) probably be solvable, but may not be,
(3) probably not be solvable, but may be, ← due to the row of zeros produced by G.E., $\vec{b}$ must be special if we do not get an equation like $0 =$ non-zero in the 3rd row
(4) certainly not be solvable.

3. (12 pts.) Suppose a 3×5 matrix A has an LU factorization with

$$L = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 2 & 3 & 1 \end{pmatrix}$$

(a) (9 pts.) Describe, in order, each of the elimination steps that were performed on A to reach the echelon form U.

1) Multiply top row by 1 and add to middle row
2) Multiply top row by -2 and add to bottom row
3) Multiply middle row by -3 and add to bottom row

(b) (3 pts.) Can you say what the rank of A must be? If so, what is it? If not, explain why not enough information has been given.

Can not say. The rank is the number of pivots, & that is only revealed by U (the outcome of elimination, not the process which L shows)

4. (21 pts. – 3 pts. each) Give short answers.

(a) For a matrix to have an inverse, it must be $n \times n$ with rank $\underline{\quad n \quad}$.

(b) The formula for the inverse of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is:

$$\frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

(c) The simple formula for the inverse of a permutation matrix P is $P^{-1} = \underline{\quad P^T \quad}$

(d) If an $m \times n$ matrix A with with $m \geq n$ is randomly chosen, its rank is virtually certain to be $\underline{\quad n \quad}$.

(e) If A is 7×13 with rank 5, then its nullspace will have dimension $\underline{\quad 8 \quad}$,

(f) ... and its columnspace will have dimension $\underline{\quad 5 \quad}$.

(g) If A is $n \times n$, and there is some $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has no solutions, then for any $\mathbf{c}$, the number of solutions to $A\mathbf{x} = \mathbf{c}$ must be one of the following: (CIRCLE ALL THAT APPLY)

none / one / infinitely many

Since $Ax=b$ has no solutions, we do not get a pivot in every row of A. Since A is square, that means we do not have a pivot in every column. From the row statement we say $Ax=c$ is not always solvable. From the column one we see that if a solution exists, there will be infinitely many.

5. (12 pts. – 4 pts. each) Give matrices M that perform the following actions on a 3×4 matrix A when the product MA is computed.

(a) reorder the rows of A, so that the top one goes to the bottom, and the others move up by one each.

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

(b) add -3 times the 2nd row to the third

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{pmatrix}$$

(c) multiply the 3rd row by -7

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -7 \end{pmatrix}$$

all must be 3×3

6. (17 pts.) Suppose you are given 5 vectors, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ in $\mathbb{R}^5$, and create a 5×5 matrix A that has the $\mathbf{v}_i$ as its columns, in order. The MATLAB command `R=rref(A)` then produces the output:

$$R = \begin{pmatrix} 1 & 2 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

(a) (8 pts.) Are the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4, \mathbf{v}_5$ independent? Explain, by referring to the *definition* of independence.

The vectors are <u>dependent</u>. To decide we must see if there is a solution to $c_1\vec{v}_1 + c_2\vec{v}_2 + \dots + c_5\vec{v}_5 = \vec{0}$ with some $c_i \neq 0$. This means we need to solve $A\vec{c} = \vec{0}$ which is equivalent to $R\vec{c} = \vec{0}$. Since this has free variables, non-zero solutions exist.

(Alternative solution: The colums of A have the same relationships as those of R. Since 2(column 1 of R) = column 2 of R we know $2\vec{v}_1 = \vec{v}_2$. Since one $\vec{v}_i$ can be expressed in terms of the others, they are dependent.

(b) (3 pts.) What is the dimension of the span of the vectors?

3 (the number of pivots in R)

(c) (6 pts.) Give a basis for the span of the vectors.

$\vec{v}_1, \vec{v}_3, \vec{v}_4$, the pivot columns from $\underline{\underline{A}}$