Math 314 Name : Key

Exam 1 February 29, 2008

1. Let

$$A = \begin{pmatrix} 1 & -2 & 0 & 2 \\ -1 & 2 & -1 & -1 \\ 2 & -4 & -1 & 5 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}.$$

(a) (10 pts.) Find all solutions to $A\mathbf{x} = \mathbf{b}$. Show all your work.

$$\left(\begin{array}{cccc|c} 1 & -2 & 0 & 2 & 1 \\ -1 & 2 & -1 & -1 & 1 \\ 2 & -4 & -1 & 5 & 4 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -2 & 0 & 2 & 1 \\ 0 & 0 & -1 & 1 & 2 \\ 0 & 0 & -1 & 1 & 2 \end{array}\right) \to \left(\begin{array}{cccc|c} 1 & -2 & 0 & 2 & 1 \\ 0 & 0 & -1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

$x - 2y + 2w = 1$

$-z + w = 2$

y, w free

$$\begin{pmatrix} x \\ y \\ z \\ w \end{pmatrix} = \begin{pmatrix} 1-2y-2w \\ y \\ -2+w \\ w \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ -2 \\ 0 \end{pmatrix} + y\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + w\begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \end{pmatrix} \quad y, w \in \mathbb{R}$$

(b) (3 pts.) Give all solutions to $A\mathbf{x} = \mathbf{0}$.

$$y\begin{pmatrix} -2 \\ 1 \\ 0 \\ 0 \end{pmatrix} + w\begin{pmatrix} -2 \\ 0 \\ 1 \\ 1 \end{pmatrix} \quad y, w \in \mathbb{R}$$

2. (10 pts.) Use elimination to find the inverse of $\begin{pmatrix} 1 & 0 & 1 \\ -1 & 2 & -1 \\ 1 & 0 & 0 \end{pmatrix}$, or show it doesn't exist. Show all your work.

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ -1 & 2 & -1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 2 & 0 & 1 & 1 & 0 \\ 0 & 0 & -1 & -1 & 0 & 1 \end{array}\right) \to \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{array}\right)$$

$$\begin{pmatrix} 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 1 & 0 & -1 \end{pmatrix}$$

3. (6 pts.) If $FG = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}$ and $G = \begin{pmatrix} -3 & 11 \\ -1 & 4 \end{pmatrix}$, what is F?

$$F = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix} G^{-1} = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix} \frac{1}{-12+11} \begin{pmatrix} 4 & -11 \\ 1 & -3 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix} \begin{pmatrix} -4 & 11 \\ -1 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -7 & 19 \\ -13 & 36 \end{pmatrix}$$

4. (10 pts.) The LU factorization of B is

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Use this factorization to solve $B\mathbf{x} = \mathbf{d}$ for $\mathbf{d} = (3, 1, 0)$. (No credit will be given for solving the system by any other method.)

Solve $L\vec{y} = \vec{d}$ and then $U\vec{x} = \vec{y}$

$L\vec{y} = \vec{d}$: $\begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 0 \end{pmatrix} \Rightarrow y_1 = 3,\ y_2 = -2,\ y_3 = 7$, so $\vec{y} = \begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix}$

$U\vec{x} = \vec{y}$: $\begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \\ 7 \end{pmatrix} \Rightarrow x_3 = 7,\ x_2 = 9,\ x_1 = -3$

so $\vec{x} = \begin{pmatrix} -3 \\ 9 \\ 7 \end{pmatrix}$

5. (3 pts.) If a matrix C has an LU factorization with

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{pmatrix},$$

describe all the elementary steps, in order, of the Gaussian elimination process performed on C.

1. Multiply top row by -2 and add it to 2^{nd} row.
2. Multiply top row by 1 and add it to 3^{rd} row.
3. Multiply ~~new~~ second row by 3 and add it to 3^{rd} row.

6. (7 pts.) Are the 3 vectors $(1, -2, 1, 0)$, $(2, 1, -3, 5)$, and $(-2, 1, 1, -3)$ linearly independent? Show your work.

Solve $x_1\begin{pmatrix} 1 \\ -2 \\ 1 \\ 0 \end{pmatrix} + x_2\begin{pmatrix} 2 \\ 1 \\ -3 \\ 5 \end{pmatrix} + x_3\begin{pmatrix} -2 \\ 1 \\ 1 \\ -3 \end{pmatrix} = \vec{0}$

$\begin{pmatrix} 1 & 2 & -2 \\ -2 & 1 & 1 \\ 1 & -3 & 1 \\ 0 & 5 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -2 \\ 0 & 5 & -3 \\ 0 & -5 & 3 \\ 0 & 5 & -3 \end{pmatrix} \to \begin{pmatrix} 1 & 2 & -2 \\ 0 & 5 & -3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Since we have a free variable,

there are non-trivial solutions, so the vectors are dependent.

7. (15 pts. – 3 pts. each) Suppose A is a $m \times n$ matrix with r pivots. Explain the relationships between r and m and/or n in each case below. (Sample answer: $r = m$, because ...). You do not need to point out that $r \leq m$ and $r \leq n$.

(a) $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions for some $\mathbf{b}$.

$r < n$, since solving leads to free variables

(b) $A\mathbf{x} = \mathbf{b}$ has no solutions for some $\mathbf{b}$, but for the $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ can be solved, there is only one solution.

$r < m$, since there are no solutions for some $\vec{b}$, so must have pivotless row

$r = n$, since there is at most one solution, so no free variables.

(c) The only solution to the homogeneous equation associated to A is the trivial one.

$r = n$, since must not have free variables on solving.

(d) The columns of A are dependent.

$r < n$, since will have free variables

(e) The solutions to $A\mathbf{x} = \mathbf{b}$ form a 2-dimensional plane.

$r = n - 2$, since must get 2 free variables.

8. (12 pts. – 3 pts. each) Give matrices with the following properties:

(a) A 4×4 matrix E, so that E will add twice the 3rd row of A to the bottom row of A when we compute EA.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 2 & 1 \end{pmatrix}$$

(b) A 3×3 matrix P, so that P will interchange the top and bottom rows of A when we compute PA

$$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

(c) A 2×2 matrix R, so that the linear transformation associated to R reflects points in the plane $\mathbb{R}^2$ about the line $y = -x$.

$\vec{e}_1 \mapsto \begin{pmatrix} 0 \\ -1 \end{pmatrix}$

$\vec{e}_2 \mapsto \begin{pmatrix} -1 \\ 0 \end{pmatrix}$

$R = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$

(d) E^{-1}, P^{-1}, and R^{-1}.

$$E^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -2 & 1 \end{pmatrix} \qquad P^{-1} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} = P \qquad R^{-1} = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} = R$$

9. (18 pts. – 3 pts. each) Are these statements True or False? Indicate T/F and explain briefly. (No points will be awarded unless an explanation is attempted.)

(a) The span of any two vectors in $\mathbb{R}^3$ forms a plane.

F. If the vectors are dependent they do not span a plane.
e.g. $\begin{pmatrix}1\\1\\1\end{pmatrix}, \begin{pmatrix}2\\2\\2\end{pmatrix}$ span a line; $\begin{pmatrix}0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\0\end{pmatrix}$ span only $\vec{0}$

(b) A system of m linear equations in n unknowns can have exactly 2 solutions.

F It may have 0, 1, or ∞-ly many solutions. The only way to have more than 1 solution is to have a free variable

(c) If A is a square singular matrix, then $A\mathbf{x} = \mathbf{b}$ cannot have any solutions.

F A singular $\Rightarrow$ A has fewer than n pivots, but it may still be possible to solve $A\vec{x} = \vec{b}$ for some $\vec{b}$. e.g. $A = \begin{pmatrix}1 & 1\\0 & 0\end{pmatrix}$ $\vec{b} = \begin{pmatrix}1\\0\end{pmatrix}$

(d) If $m < n$, then n vectors in $\mathbb{R}^m$ must be linearly dependent.

T Placing the n vectors into the columns of a $m \times n$ matrix, we cannot get a pivot in every column. Thus the vectors must be dependent.

(e) If A is $n \times n$ and non-singular, then the columns of A span $\mathbb{R}^n$.

T A non-singular $\Rightarrow$ A has n pivots so $A\vec{x} = \vec{b}$ can be solved for every $\vec{b}$
OR A non-singular $\Rightarrow$ $A\vec{x} = \vec{b}$ has solution $\vec{x} = A^{-1}\vec{b}$, so can solve for every $\vec{b}$

(f) If $A\mathbf{x} = \mathbf{b}$ has exactly one solution for a particular $\mathbf{b} \in \mathbb{R}^m$, then $A\mathbf{x} = \mathbf{c}$ has exactly one solution for all $\mathbf{c} \in \mathbb{R}^m$.

F A will have a pivot in every column, but not necessarily in every row.
e.g. $A = \begin{pmatrix}1 & 1\\0 & 1\\0 & 0\end{pmatrix}$ $\vec{b} = \begin{pmatrix}1\\0\\0\end{pmatrix}$, $\vec{c} = \begin{pmatrix}0\\0\\1\end{pmatrix}$. $A\vec{x} = \vec{b}$ has only 1 solution, but $A\vec{x} = \vec{c}$ has none

10. (6 pts.) Suppose a linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^5$ is one-to-one. Must T also be onto? Explain. (Hint: What can you say about the matrix A such that $T = T_A$?)

T must be onto. Since T is 1-1, $A\vec{x} = \vec{b}$ has at most one solution for each $\vec{b}$, hence A has a pivot in every column. But A is 5×5, so A has a pivot in every row. But this means $A\vec{x} = \vec{b}$ is solvable for every $\vec{b}$. Thus T is onto.