Math 314 | Name : ____________________
Exam 1 | February 29, 2008

1. Let

$$A = \begin{pmatrix} 1 & -2 & 0 & 2 \\ -1 & 2 & -1 & -1 \\ 2 & -4 & -1 & 5 \end{pmatrix}, \quad \mathbf{b} = \begin{pmatrix} 1 \\ 1 \\ 4 \end{pmatrix}.$$

   (a) (10 pts.) Find all solutions to $A\mathbf{x} = \mathbf{b}$. Show all your work.

   (b) (3 pts.) Give all solutions to $A\mathbf{x} = \mathbf{0}$.

2. (10 pts.) Use elimination to find the inverse of $\begin{pmatrix} 1 & 0 & 1 \\ -1 & 2 & -1 \\ 1 & 0 & 0 \end{pmatrix}$, or show it doesn't exist. Show all your work.

3. (6 pts.) If $FG = \begin{pmatrix} 2 & -1 \\ 3 & 1 \end{pmatrix}$ and $G = \begin{pmatrix} -3 & 11 \\ -1 & 4 \end{pmatrix}$, what is F?

4. (10 pts.) The LU factorization of B is

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ -1 & 2 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}.$$

Use this factorization to solve $B\mathbf{x} = \mathbf{d}$ for $\mathbf{d} = (3, 1, 0)$. (No credit will be given for solving the system by any other method.)

5. (3 pts.) If a matrix C has an LU factorization with

$$L = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & -3 & 1 \end{pmatrix},$$

describe all the elementary steps, in order, of the Gaussian elimination process performed on C.

6. (7 pts.) Are the 3 vectors $(1, -2, 1, 0)$, $(2, 1, -3, 5)$, and $(-2, 1, 1, -3)$ linearly independent? Show your work.

7. (15 pts. – 3 pts. each) Suppose A is a $m \times n$ matrix with r pivots. Explain the relationships between r and m and/or n in each case below. (Sample answer: $r = m$, because ...). You do not need to point out that $r \leq m$ and $r \leq n$.

   (a) $A\mathbf{x} = \mathbf{b}$ has infinitely many solutions for some $\mathbf{b}$.

   (b) $A\mathbf{x} = \mathbf{b}$ has no solutions for some $\mathbf{b}$, but for the $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ can be solved, there is only one solution.

   (c) The only solution to the homogeneous equation associated to A is the trivial one. (The homogeneous equation means $A\mathbf{x} = \mathbf{0}$.)

   (d) The columns of A are dependent.

   (e) The solutions to $A\mathbf{x} = \mathbf{b}$ form a 2-dimensional plane.

8. (12 pts. – 3 pts. each) Give matrices with the following properties:

   (a) A 4×4 matrix E, so that E will add twice the 3rd row of A to the bottom row of A when we compute EA.

   (b) A 3×3 matrix P, so that P will interchange the top and bottom rows of A when we compute PA

   (c) A 2×2 matrix R, so that the linear transformation associated to R reflects points in the plane $\mathbb{R}^2$ about the line $y = -x$. (This question is not appropriate (yet) for Spring 2013)

   (d) E^{-1}, P^{-1}, and R^{-1}.

9. (18 pts. – 3 pts. each) Are these statements True or False? Indicate T/F and explain briefly. (No points will be awarded unless an explanation is attempted.)

   (a) The span of any two vectors in $\mathbb{R}^3$ forms a plane.

   (b) A system of m linear equations in n unknowns can have exactly 2 solutions.

   (c) If A is a square singular matrix, then $A\mathbf{x} = \mathbf{b}$ cannot have any solutions.

   (d) If $m < n$, then n vectors in $\mathbb{R}^m$ must be linearly dependent.

   (e) If A is $n \times n$ and non-singular, then the columns of A span $\mathbb{R}^n$.

   (f) If $A\mathbf{x} = \mathbf{b}$ has exactly one solution for a particular $\mathbf{b} \in \mathbb{R}^m$, then $A\mathbf{x} = \mathbf{c}$ has exactly one solution for all $\mathbf{c} \in \mathbb{R}^m$.

10. (6 pts.) Suppose a linear transformation $T : \mathbb{R}^5 \to \mathbb{R}^5$ is one-to-one. Must T also be onto? Explain. (Hint: What can you say about the matrix A such that $T = T_A$?) (This question is not appropriate (yet) for Spring 2013)