Math F251: Section 3.5 Worksheet Solutions

1. Find dy/dx by implicit differentiation.

$y \cos x = x^2 + y^2$

$$\frac{dy}{dx}\cos x - y\sin x = 2x + 2y\frac{dy}{dx}$$

$$\frac{dy}{dx}(\cos x - 2y) = 2x + y\sin x$$

$$\frac{dy}{dx} = \frac{2x + y\sin x}{\cos x - 2y}$$

2. Consider the equation

$$\sqrt{x} + \sqrt{y} = 1 \qquad (*)$$

(a) Find y' by implicit differentiation.

$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}\frac{dy}{dx} = 0$$

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{\sqrt{y}}{\sqrt{x}}$$

(b) Solve $(*)$ explicitly for y and differentiate to get y' in terms of x.

$$\sqrt{y} = 1 - \sqrt{x}$$

$$y = (1 - \sqrt{x})^2$$

$$\frac{dy}{dx} = 2(1-\sqrt{x})\left(-\frac{1}{2}x^{-1/2}\right) = -\frac{1-\sqrt{x}}{\sqrt{x}}$$

(c) Check that your solutions in (a) and (b) are consistent.

Since $\sqrt{y} = 1 - \sqrt{x}$ these agree.

3. (*A §3.4 question.*) For what values of r does the function $y = e^{rt}$ satisfy the differential equation $y'' - 4y' + y = 0$?

$y' = re^{rt}$

$y'' = r^2 e^{rt}$

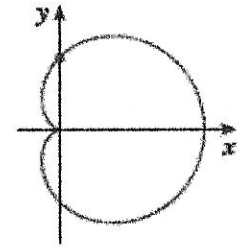
$$y'' - 4y' + y = r^2 e^{rt} - 4re^{rt} + e^{rt}$$
$$= e^{rt}(r^2 - 4r + 1)$$

This is 0 exactly when $r^2 - 4r + 1 = 0$ (since $e^{rt} \neq 0$)

$$r = \frac{-4 \pm \sqrt{16-4}}{2} = -2 \pm \sqrt{3}$$

4. For the "cardiod" shown, with the equation and point given, find an equation of the tangent line.

$x^2 + y^2 = (2x^2 + 2y^2 - x)^2, \qquad (0, \frac{1}{2})$



$$2x + 2yy' = 2(2x^2 + 2y^2 - x)(4x + 4yy' - 1)$$

Letting $x = 0$, $y = \frac{1}{2}$

$$y' = 2(\tfrac{1}{2})(2y' - 1)$$
$$y' = 2y' - 1$$
$$y' = 1$$

So $y - \frac{1}{2} = 1(x - 0)$

$$y = \frac{1}{2} + x$$

5. If $xy + e^y = e$, find the value of y'' at the point where $x = 0$.

$$1 \cdot y + xy' + e^y y' = 0$$
$$y' + (1 \cdot y' + xy'') + e^y (y')^2 + e^y y'' = 0$$

When $x = 0$ $\quad e^y = e \Rightarrow y = 1$

$$1 \cdot 1 + 0y' + ey' = 0 \Rightarrow y' = -\frac{1}{e}$$
$$-\frac{1}{e} + (-\frac{1}{e} + 0) + e(-\frac{1}{e})^2 + ey'' = 0$$

$$\to -\frac{1}{e} + ey'' = 0$$
$$ey'' = \frac{1}{e}$$
$$y'' = \frac{1}{e^2}$$