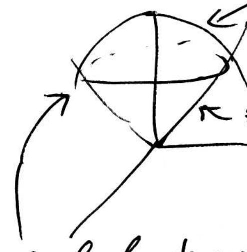
Exam 2
Math 202 F01

Name: Solutions
11/12/12

1. (12 pts.) A 3-d object is bounded below by $z = \sqrt{x^2+y^2}$ and above by $z = 6-x^2-y^2$. Its mass density is given by $\rho(x,y,z) = x^2+y^2+z^2$. Using CYLINDRICAL COORDINATES, set up an appropriate integral expression for $\bar{z}$, the z-coordinate of its center of mass. DO NOT evaluate the integrals.

$z = 6-x^2-y^2$
$= 6-r^2$



$z = \sqrt{x^2+y^2} = r$

Circle of intersection
$z = r$ + $z = 6-r^2$
$r = 6-r^2$
$r^2+r-6=0$
$(r+3)(r-2)=0$
$r = -3, 2$

$$\bar{z} = \frac{\iiint_Q z\rho\, dV}{\iiint_Q \rho\, dV}$$

$\rho = x^2+y^2+z^2$
$= r^2+z^2$

$$= \frac{\int_0^{2\pi}\int_0^2\int_r^{6-r^2} z(r^2+z^2)\, r\, dz\, dr\, d\theta}{\int_0^{2\pi}\int_0^2\int_r^{6-r^2} (r^2+z^2)\, r\, dz\, dr\, d\theta}$$

2. (12 pts.) Consider the function $f(x,y) = x^3+y^3-3xy+12$.

(a) (4 pts.) Show that $(0,0)$ and $(1,1)$ are critical points of f. (They are actually the only critical points of f, but you need not show that.)

$\nabla f = \langle 3x^2-3y, 3y^2-x^2 \rangle$

so $\nabla f(0,0) = \langle 0, 0 \rangle$
$\nabla f(1,1) = \langle 0, 0 \rangle$

(b) (8 pts.) Apply the 2nd derivative test at each of these points, and state your conclusions from it.

$$D = \begin{vmatrix} f_{xx} & f_{xy} \\ f_{xy} & f_{yy} \end{vmatrix} = \begin{vmatrix} 6x & -3 \\ -3 & 6y \end{vmatrix} = 36xy-9$$

$D(0,0) = -9 < 0 \Rightarrow$ (0,0) is a saddle

$$\left.\begin{array}{l} D(1,1) = 36-9 = 27 > 0 \\ f_{xx}(1,1) = 6 > 0 \end{array}\right\} \Rightarrow \text{(1,1) is a local minimum}$$

3. (7 pts.) Give an equation for the tangent plane to the surface $\underbrace{x^2 - 2y^2 + z^2 + yz}_{f(x,y,z)} = 2$ at the point $(2, 1, -1)$.

$\nabla f = \langle 2x, -4y+z, 2z+y \rangle$

$\nabla f(2,1,-1) = \langle 4, -5, -1 \rangle$

$$4(x-2) - 5(y-1) - 1(z--1) = 0$$

$$4x - 5y - z = 4$$

4. (15 pts.–5 pts. each) The temperature at a point (x, y, z) is given by

$$T(x, y, z) = 200e^{-x^2-3y^2-9z^2},$$

where T is measured in $^\circ C$, and x, y, z are measured in meters.

(a) Find the rate of change of the temperature at the point $(2, -1, 2)$ in the direction towards the point $(3, -3, 3)$. GIVE UNITS.

$\vec{v} = (3,-3,3) - (2,-1,2) = \langle 1, -2, 1 \rangle$

$\vec{u} = \frac{\vec{v}}{\|\vec{v}\|} = \frac{1}{\sqrt{6}} \langle 1, -2, 1 \rangle$

$\nabla T(2,-1,2) = 200e^{-x^2-3y^2-9z^2} \langle -2x, -6y, -18z \rangle \Big|_{(2,-1,2)} = 200e^{-43} \langle -4, 6, -36 \rangle$

So $D_{\vec{u}} T = \nabla T \cdot \vec{u} = 200e^{-43} \frac{1}{\sqrt{6}} \left((-4)(1) + 6(-2) + -36(1) \right) = \frac{200e^{-43}(-52)}{\sqrt{6}}$

$= -\frac{5200\sqrt{2}e^{-43}}{\sqrt{3}}$ $^\circ/m$

(b) At $(2, -1, 2)$, in what direction does T increase most rapidly?

Same direction as $\nabla T(2,1,-2)$

So $\langle -4, 6, -36 \rangle$ or $\langle -2, 3, -18 \rangle$

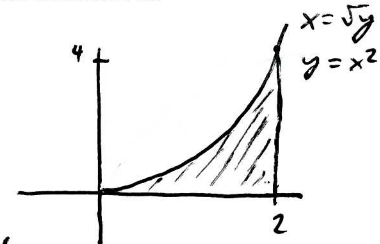
(c) What is the maximum rate of change of T at $(2, -1, 2)$, among all directions?

$\| \nabla T(2,1,-2) \| = 200e^{-43} \| \langle -4, 6, -36 \rangle \| = 200e^{-43} \sqrt{4^2 + 6^2 + 36^2}$

$= 200e^{-43}(2)\sqrt{1 + 3^2 + 18^2} = 400e^{-43}\sqrt{337}$

5. (12 pts.) Reverse the order of integration in the following integral, and then evaluate it.

$$\int_0^4 \int_{\sqrt{y}}^2 \sqrt{x^3+1}\,dx\,dy$$



$$= \int_0^2 \int_0^{x^2} \sqrt{x^3+1}\,dy\,dx = \int_0^2 \left(\sqrt{x^3+1}\,y\right)\Big|_{y=0}^{x^2} dx$$

$$= \int_0^2 x^2\sqrt{x^3+1}\,dx = \frac{1}{3}\int_1^9 u^{1/2}\,du = \frac{1}{3}\left(\frac{2}{3}\right)u^{3/2}\Big|_{u=1}^{9} = \frac{2}{9}\left(9^{3/2}-1^{3/2}\right)$$

$u = x^3+1$

$du = 3x^2\,dx$

$\frac{1}{3}du = x^2\,dx$

$$= \frac{2}{9}(27-1) = \frac{52}{9}$$

6. (10 pts.) Ohm's law states that in an electrical circuit the current, I, depends on the voltage, V, and resistance, R, by

$$I = V/R.$$

Suppose at some moment $R = 100$ ohms, $V = 32$ volts, $dR/dt = 0.03$ ohms/s, and $dV/dt = -0.01$ volts/s. Determine dI/dt at that moment. GIVE UNITS. (Hint: Use the multivariable chain rule. The unit 'volt/ohm' is also called an 'ampere'.)

$$\frac{dI}{dt} = \frac{\partial I}{\partial V}\frac{dV}{dt} + \frac{\partial I}{\partial R}\frac{dR}{dt}$$

$$= \frac{1}{R}\frac{dV}{dt} + \left(\frac{-V}{R^2}\right)\frac{dR}{dt}$$

$$= \left(\frac{1}{100\text{ ohms}}\right)\left(-.01\,\frac{\text{volts}}{\text{s}}\right) + \left(\frac{-32\text{ volts}}{100^2\text{ ohms}^2}\right)\left(.03\,\frac{\text{ohms}}{\text{s}}\right)$$

$$= -.0001\,\frac{\text{volts}}{\text{ohm}\cdot\text{s}} + -\frac{.96}{10^4}\,\frac{\text{volts}}{\text{ohm}\cdot\text{sec}}$$

$$= -.000196\,\frac{\text{amperes}}{\text{s}}$$